On The Solution of Existence of Nonlinear Integral-and Integrodifferential Equations

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Abstract

In this paper we study the existence and uniqueness for mixed Volterra – Fredholm integral and integrodifferential equations By using the extensions of Banach's contraction principle in complete cone metric space.

Keywords: Fixed point theorem; Cone metric space; Comparison function; Banach space.
Introduction

The purpose of this paper is to study the existence of solution for Volterra – Fredholm integral equation of the form:

\[ x(t) = f(t, x(t)) + \int_{0}^{t} k(t, s)k_1(s, x(s))ds + \int_{0}^{b} h(t, s)h_1(s, x(s))ds, \quad t \in J = [0, b] \] (1)

and is study the existence of solution for the integro differential Volterra-Fredholm integral equation of the first order of the form:

\[ x'(t) = f(t, x(t)) + \int_{0}^{t} k(t, s)k_1(s, x(s))ds + \int_{0}^{b} h(t, s)h_1(s, x(s))ds \] ..............(2)

\[ x(0) = x_0 \] ..............(3)

Where \( f : J \times Z \rightarrow Z, k : J \times J \rightarrow J, k_1, h_1 : J \times Z \rightarrow Z \), are continuous and the

given \( x_0 \) is element of \( Z \), \( Z \) is a Banach space with norm \( \| \cdot \| \)

The integrodiffirential equations and integral equations have been studied by many authors [2,4,5,6,7]. Tidke [7] studied the existence of solution for Volterra – Fredholm integro differential equations of second order of the form:

\[ x''(t) = Ax(t) + \int_{0}^{t} k(t, s, x(s))ks + \int_{0}^{b} h(t, s, x(s))ds \]

And B.G.Pachpatte [6] studied the existence of solution for Volterra integral equation in two variables of the form

\[ u(x, y) = f(x, y, u(x, y), K(x, y)) \]

\[ (Ku)(x, y) = \int_{a}^{x} \int_{b}^{y} K(x, y, m, n, u(m, n))dm \ dn \]

Finally in [4], Balachandran and Kim established sufficient conditions for the existence and uniqueness of random solutions of nonlinear Volterra-Fredholm stochastic integral equations of mixed type by using admissibility theory and fixed point theorem.
**Preliminaries**

Let us recall the concepts of the cone metric space and we refer the reader for the more details. Let $E$ be a real Banach space and $P$ is a subset of $E$. Then $P$ is called a cone if and only if,

1. $P$ is closed, nonempty and $P \neq \{0\}$;

2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

3. $x \in P$ and $-x \in P \Rightarrow x = 0$

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of $P$. The cone $P$ is called normal if there is a number $K > 0$ such that implies $0 \leq x \leq y$

The least Positive number satisfying $x, y \in E$, for every $\|x\| \leq K \|y\|$

above is called the normal constant of $P$

In the following way, we always suppose $E$ is a real Banach space, $P$ is a cone in $E$ with $\text{int} P \neq \emptyset$, and $\leq$ is a partial ordering with respect to $P$.

**Definition 1:**[7]

Let $X$ be nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$;

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called cone metric space. The concept of cone metric space is more general than that of metric space.
Definition 2:[7]

Let X be an ordered space. A function \( \phi : X \rightarrow X \) is said to a comparison function if for every \( x, y \in X, x \leq y \) implies that \( \phi(x) \leq \phi(y), \phi(x) \leq x \) and
\[
\lim_{n \to \infty} \|\phi^n(x)\| = 0, \text{ for every } x \in X
\]

Definition 3:[8]

The function \( x \in B \) given by
\[
x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t k(t, \tau)k_1(\tau, x(\tau))d\tau + \int_0^b h(t, \tau)h_1(\tau, x(\tau))d\tau]ds \quad \ldots \quad (4)
\]

Is called the solution of the initial value problem (2)-(3).

We need the following Lemma for further discussion.

Lemma :[8]

Let \((X,d)\) be a complete cone metric space, where \( P \) is a normal cone with normal constant \( K \). Let be a function such that there exist a \( f : X \rightarrow X \) comparison function \( \phi : P \rightarrow P \)

such that \( d(f(x), f(y)) \leq \phi(d(x, y)) \) for every \( x, y \in X \). Then \( f \) has a unique fixed point.

- We list the following hypotheses for our convenience:

(H1) Ther exist continuous \( P_1, P_2 : J \times J \rightarrow R^+ \) and a comparison function \( \phi : R^2 \rightarrow R^2 \) such that

\[
k(t, s)[k_1(s, u) - k_1(s, v)], \alpha[k_1(s, u) - k_1(s, v)] \leq M P_1(s)\phi(d(u, v)),
h(t, s)[h_1(s, u) - h_1(s, v)], \alpha[h_1(s, u) - h_1(s, v)] \leq N P_2(s)\phi(d(u, v)),
\]

and

\[
[f(s, u) - f(s, v)], \alpha[f(s, u) - f(s, v)] \leq r\phi(d(u, v)), \quad -1 < r \leq 0
\]
For every $t, s \in J$ and $u, v \in Z$

(H2) $\sup_{s \in J} \int_{0}^{b} [M P_1(s) + N P_2(s)] ds \leq 1$

(H3) $\int_{0}^{b} \int_{0}^{b} [M P_1(s) + N P_2(s)] ds dt \leq 1$

(H4) $\int_{0}^{b} ds \leq 1$

**Main Results**

In this section we will prove the main result of this paper:

**Theorem 1:** Assume that hypotheses (H1)-(H2) hold then the integral equation (1) has unique solution $x$ on $J$.

**Proof:**

The operator $F : B \to B$ is defined by:

$$F(x)(t) = f(t, x(t)) dt + \int_{0}^{t} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds, \quad t \in J \quad (5)$$

By using the hypotheses H1 – H2, we have
\[
\left\| F_x(t) - F_y(t) \right\| \leq \left\| f(t, x(t)) + \int_{0}^{1} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds - f(t, y(t)) \right\| \\
\leq \left( \int_{0}^{t} k(t, s) k_1(s, y(s)) ds - \int_{0}^{b} h(t, s) h_1(s, y(s)) ds \right \|, \alpha \left\| f(t, x(t)) + \int_{0}^{b} h(t, s) h_1(s, y(s)) ds \right\| + \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds - f(t, y(t)) \right) \\
\leq \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds - \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right \| + \alpha \left\| f(t, x(t)) + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right\| + \alpha \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds - f(t, y(t)) \right) \\
\leq \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds - \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right \| + \alpha \left\| f(t, x(t)) + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right\| + \alpha \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds - f(t, y(t)) \right) \\
\leq \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds - \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right \| + \alpha \left\| f(t, x(t)) + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds \right\| + \alpha \left( \int_{0}^{t} k(t, s) k_1(s, x(s)) ds + \int_{0}^{b} h(t, s) h_1(s, x(s)) ds - f(t, y(t)) \right) \\
\leq r \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) + \int_{0}^{t} M_{P_1}(s) \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) ds + \int_{0}^{b} N_{P_2}(s) \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) ds \\
\leq r \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) + \int_{0}^{t} M_{P_1}(s) \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) ds + \int_{0}^{b} N_{P_2}(s) \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) ds \\
\leq r \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) + \int_{0}^{b} [M_{P_1}(s) + N_{P_2}(s)] \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) ds \\
\leq r \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) + \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) \\
= (r + 1) \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) \\
\leq \phi\left(\|x - y\|_{\infty}, \alpha \|x - y\|_{\infty}\right) \quad , \quad \| + 1 \leq 1 
\]

This implies that \( d(Fx, Fy) \leq \phi(d(x, y)) \) for every \( x, y \in B \). Now an application of lemma 1, the operator has a unique point in \( B \). This means that the equation (1) has unique solution. This completes the proof of theorem (1).
Theorem 2: Assume that hypotheses (H1)-(H3) and (H4) hold then the initial value problem (2)-(3) has unique solution $x$ on $J$.

Proof:

The operator $G : B \rightarrow B$ is defined by

$$
Gx(t) = x_0 + \int_0^t f(s, x(s)) ds + \int_0^t \int_k(t, \tau)k_1(\tau, x(\tau)) d\tau + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) d\tau ds, \ t \in J \ldots \ldots (6)
$$

By using the hypotheses H1,H3,H4, we have.

$$
\|Gx(t) - Gy(t)\|, \alpha \|Gx(t) - Gy(t)\|
\leq \left( \left\| \int_0^t f(s, x(s)) ds + \int_0^t \int_k(t, \tau)k_1(\tau, x(\tau)) d\tau + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) d\tau ds \right\|, \alpha \left\| \int_0^t f(s, x(s)) ds + \int_0^t \int_k(t, \tau)k_1(\tau, x(\tau)) d\tau + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) d\tau ds \right\| \right)
$$
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\[ \leq \left( \int_0^b \left\| f(s, x(s)) - f(s, y(s)) \right\| ds + \int_0^b \left[ \int_0^b k(t, \tau) \left\| k_i(\tau, x(\tau)) - k_i(\tau, y(\tau)) \right\| d\tau \right] ds \right) \]

\[ + \alpha \int_0^b \int_0^b h(t, \tau) \left\| h_i(\tau, x(\tau)) - h_i(\tau, y(\tau)) \right\| d\tau ds + \alpha \int_0^b \int_0^b k(t, \tau) \left\| k_i(\tau, x(\tau)) - k_i(\tau, y(\tau)) \right\| d\tau ds \]

\[ \leq \left( \int_0^b \left\| f(s, x(s)) - f(s, y(s)) \right\| ds + \alpha \int_0^b \int_0^b k(t, \tau) \left\| k_i(\tau, x(\tau)) - k_i(\tau, y(\tau)) \right\| d\tau ds \right) \]

\[ + \alpha \int_0^b \int_0^b h(t, \tau) \left\| h_i(\tau, x(\tau)) - h_i(\tau, y(\tau)) \right\| d\tau ds \]

This implies that \( d(Fx, Fy) \leq \phi(d(x, y)) \) for every \( x, y \in B \)

Now an application of lemma 1 the operator has a unique point in \( B \). This means that the equation (1) – (2) has unique solution. This couplets the proof of theorem (2).

Application

In this section we give an example to illustrate the usefulness of our result. In equation (1), we define:
k(t,s) = s, k_1(s, x(s)) = \frac{x}{2}, h(t,s) = s, h_1(s, x(s)) = \frac{x^2}{2}, f(s, x(s)) = \lambda x + s \\
\text{where } t, s \in [0,1], x \in C([0,1], \mathbb{R}), \lambda \text{ constant}

And metric \( d(x,y) = (\|x-y\|_\infty, \alpha \|x-y\|_\infty) \) on \( C([0,1], \mathbb{R}) \) and \( \alpha \geq 0 \). Then clearly \( C([0,1], \mathbb{R}) \) is complete cone metric space.

Now we have

\[
(\|k(t,s)[k_1(s, x(s)) - k_1(s, y(s))]\|, \alpha \|k(t,s)[k_1(s, x(s)) - k_1(s, y(s))]\|)
\]

\[
= (s\left|\frac{x}{2} - \frac{y}{2}\right|, \alpha s\left|\frac{x}{2} - \frac{y}{2}\right|)
\]

\[
= (\frac{s}{2}|x-y|, \alpha \frac{s}{2}|x-y|)
\]

\[
\leq \frac{s}{2}(\|x-y\|_\infty, \alpha \|x-y\|_\infty)
\]

\[
= M P_1^* \Phi^* (\|x-y\|_\infty, \alpha \|x-y\|_\infty),
\]

Where \( P_1^*(s) = s \), which is continuous function of \( [0,1] \times [0,1] \) into \( \mathbb{R}^+ \) and a comparison function \( \Phi^*: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( \Phi^*(x,y) = (x,y) \). Similarly, we can show that
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\[ \{ h(t,s)h_1(s,x(s))h_1(s,y(s)) \| \alpha h(t,s)h_1(s,x(s))h_1(s,y(s)) \| \leq P_2 \phi (x-y,\alpha x-y \| u \| ) \],

where \( P_2(s) = s \), which is continuous function of \([0,1]\times [0,1]\] into \( R^+ \). Similarly, we can show that

\[ (f(s,x(s)) - f(s,y(s))) \| \alpha (f(s,x(s)) - f(s,y(s))) \| \leq \lambda \phi (x-y,\alpha x-y \| u \| ), \]

\(-1 < \lambda \leq 0 \)

Moreover,

\[ \int_0^1 [M P_1(s) + N P_2(s)]ds = \int_0^1 [\frac{1}{2} s + \frac{1}{2} s]ds = \frac{1}{2} < 1 \]

Also

\[ \int_0^1 [M P_1(s) + M P_2(s)]dsdt = \int_0^1 [\frac{1}{2} s + \frac{1}{2} s]dsdt = \frac{1}{2} dt = \frac{1}{2} < 1 \]

Also

\[ \int_0^1 ds = 1 \]

With these choices of function, all requirements of Theorem 1. And Theorem 2 are satisfied. Hence the existence and uniqueness are verified.

References


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