On The Solution of Existence of Nonlinear Integral-and Integrodifferential Equations

Noora L. Husein
Department of Mathematics \ College of Education
University of Mosul

Received 02/04/2013
Accepted 05/06/2013

Aabstract

In this paper we study the existence and uniqueness for mixed Volterra – Fredholm integral and integrodifferential equations By using the extensions of Banach's contraction principle in complete cone metric space.

Keywords : Fixed point theorem ; Cone metric space ; Comparison function ; Banach space.
Introduction

The purpose of this paper is to study the existence of solution for Volterra–Fredholm integral equation of the form:

\[ x(t) = f(t, x(t)) + \int_{0}^{t} k(t, s)k_{1}(s, x(s))ds + \int_{0}^{b} h(t, s)h_{1}(s, x(s))ds, \quad t \in J = [0, b] \quad \ldots \quad (1) \]

and is study the existence of solution for the integro differential Volterra-Fredholm integral equation of the first order of the form:

\[ x'(t) = f(t, x(t)) + \int_{0}^{t} k(t, s)k_{1}(s, x(s))ds + \int_{0}^{b} h(t, s)h_{1}(s, x(s))ds \quad \ldots \ldots \ldots \ldots (2) \]

\[ x(0) = x_{0} \quad \ldots \ldots \ldots \ldots (3) \]

Where \( f : J \times Z \rightarrow Z, k, h : J \times J \rightarrow J, k_{1}, h_{1} : J \times Z \rightarrow Z \), are continuous and the given \( x_{0} \) is element of \( Z \), \( Z \) is a Banach space with norm \( \| \cdot \| \).

The integrodifferential equations and integral equations have been studied by many authors [2,4,5,6,7]. Tidke [7] studied the existence of solution for Volterra–Fredholm integro differential equations of second order of the form:

\[ x''(t) = Ax(t) + \int_{0}^{t} k(t, s, x(s))kds + \int_{0}^{b} h(t, s, x(s))ds \]

And B.G.Pachpatte [6] studied the existence of solution for Volterra integral equation in two variables of the form

\[ u(x, y) = f(x, y, u(x, y), K(x, y)) \]

\[ (Ku)(x, y) = \int_{a}^{x} \int_{b}^{y} K(x, y, m, n, u(m, n))dn dm \]

Finally in [4], Balachandran and Kim established sufficient conditions for the existence and uniqueness of random solutions of
nonlinear Volterra-Fredholm stochastic integral equations of mixed type by using admissibility theory and fixed point theorem.

**Preliminaries**

Let us recall the concepts of the cone metric space and we refer the reader for the more details. Let $E$ be a real Banach space and $P$ is a subset of $E$. Then $P$ is called a cone if and only if,

1. $P$ is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of $P$. The cone $P$ is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of $P$.

In the following way, we always suppose $E$ is a real Banach space, $P$ is a cone in $E$ with $\text{int}P \neq \emptyset$, and $\leq$ is a partial ordering with respect to $P$.

**Definition 1:**[7]

Let $X$ be nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$;
Then d is called a cone metric on X and (X,d) is called cone metric space. The concept of cone metric space is more general than that of metric space.

**Definition 2:**[7]  
Let X be an ordered space. A function \( \phi : X \rightarrow X \) is said to a comparison function if for every \( x, y \in X, x \leq y \) implies that \( \phi(x) \leq \phi(y), \phi(x) \leq x \) and \( \lim_{n \to \infty} \|\phi^n(x)\| = 0 \), for every \( x \in X \).

**Definition 3:**[8]  
The function \( x \in B \) given by  
\[
x(t) = x_0 + \int_0^t f(s, x(s))ds + \int_0^t \int_0^s k(t, \tau)k_1(\tau, x(\tau))d\tau + \int_0^t h(t, \tau)h_1(\tau, x(\tau))d\tau ds ......(4)
\]

Is called the solution of the initial value problem (2)-(3).

We need the following Lemma for further discussion.

**Lemma:**[8]  
Let \( (X, d) \) be a complete cone metric space, where \( P \) is a normal cone with normal constant \( K \). Let \( f : X \rightarrow X \) be a function such that there exist a comparison function \( \phi : P \rightarrow P \) such that \( d(f(x), f(y)) \leq \phi(d(x, y)) \) for every \( x, y \in X \). Then f has a unique fixed point.

- We list the following hypotheses for our convenience:
  (H1) There exist continuous \( P_1, P_2 : J \times J \rightarrow R^+ \) and a comparison function \( \phi : R^2 \rightarrow R^2 \) such that
For every $t, s \in J$ and $u, v \in Z$

(H2) $\sup_{s \in J} \int_{0}^{b} [M P_1(s) + N P_2(s)] ds \leq 1$

(H3) $\int_{0}^{b} \int_{0}^{b} [M P_1(s) + N P_2(s)] ds dt \leq 1$

(H4) $\int_{0}^{b} ds \leq 1$

**Main Results**

In this section we will prove the main result of this paper:

**Theorem 1:** Assume that hypotheses (H1)-(H2) hold then the integral equation (1) has unique solution $x$ on $J$.

**Proof:**

The operator $F : B \to B$ is defined by:
\[ F_x(t) = \int_0^t f(t,x(t))dt + \int_0^t k(t,s)k_1(s,x(s))ds + \int_0^b h(t,s)h_1(s,x(s))ds, \quad t \in J \quad \text{......(5)} \]

By using the hypotheses H1 – H2, we have

\[
\left( \|F_x(t) - F_y(t)\|, \alpha \|F_x(t) - F_y(t)\| \right)
\]

\[
\leq \left( \left\| \int_0^t f(t,x(t))dt + \int_0^t k(t,s)k_1(s,x(s))ds + \int_0^b h(t,s)h_1(s,x(s))ds - f(t,y(t)) \right\| \right.
\]

\[
- \int_0^t k(t,s)k_1(s,y(s))ds - \int_0^b h(t,s)h_1(s,y(s))ds \bigg\|, \alpha \bigg\| f(t,x(t)) +
\]

\[
+ \int_0^t k(t,s)k_1(s,x(s))ds + \int_0^b h(t,s)h_1(s,x(s))ds - f(t,y(t)) \bigg\|
\]

\[
- \int_0^t k(t,s)k_1(s,y(s))ds - \int_0^b h(t,s)h_1(s,y(s))ds \bigg\|
\]

\[
\leq \left( \left\| \int_0^t f(t,x(t))dt - f(t,y(t)) \right\| + \int_0^t \left\| k(t,s)[k_1(s,x(s)) - k_1(s,y(s))] \right\| ds \right.
\]

\[
+ \int_0^b \left\| h(s,t)[h_1(s,x(s)) - h_1(s,y(s))] \right\| ds, \alpha \|f(t,x(t)) - f(t,y(t))\| 
\]

\[
+ \alpha \int_0^t \left\| k(s,t)[k_1(s,x(s)) - k_1(s,y(s))] \right\| ds + \alpha \int_0^b \left\| h(s,t)[h_1(s,x(s)) - h_1(s,y(s))] \right\| ds
\]

\[
\leq \left( \left\| f(t,x(t)) - f(t,y(t)) \right\|, \alpha \|f(t,x(t)) - f(t,y(t))\| \right) + \left( \int_0^t \left\| k(s,t)[k_1(s,x(s)) - k_1(s,y(s))] \right\| ds \right.
\]

\[
- \alpha \int_0^t \left\| k(s,t)[k_1(s,x(s)) - k_1(s,y(s))] \right\| ds + \left( \int_0^b \left\| h(s,t)[h_1(s,x(s)) - h_1(s,y(s))] \right\| ds \right.
\]

\[
- \alpha \int_0^b \left\| h(s,t)[h_1(s,x(s)) - h_1(s,y(s))] \right\| ds
\]

\[
\leq \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha) + \int_0^t M_P(s)\phi(\|x - y\|_\omega, \alpha\|x - y\|_\omega)ds + \int_0^b N_P(s)\phi(\|x - y\|_\omega, \alpha\|x - y\|_\omega)ds
\]

\[
\leq \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha) + \int_0^t M_P(s)\phi(\|x - y\|_\omega, \alpha\|x - y\|_\omega)ds + \int_0^b N_P(s)\phi(\|x - y\|_\omega, \alpha\|x - y\|_\omega)ds
\]

\[
\leq \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha) + \int_0^t (M_P(s) + N_P(s))\phi(\|x - y\|_\omega, \alpha\|x - y\|_\omega)ds
\]

\[
\leq \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha)
\]

\[
= (r + 1) \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha)
\]

\[
\leq \phi(\|x - y\|_\alpha, \alpha\|x - y\|_\alpha) \quad , \quad |r + 1| \leq 1
\]
This implies that \( d(Fx, Fy) \leq \phi(d(x, y)) \) for every \( x, y \in B \). Now an application of lemma 1, the operator has a unique point in \( B \). This means that the equation (1) has unique solution. This completes the proof of theorem (1).

**Theorem 2:** Assume that hypotheses (H1)-(H3) and (H4) hold then the initial value problem (2)-(3) has unique solution \( x \) on \( J \).

**Proof:**

The operator \( G : B \to B \) is defined by

\[
Gx(t) = x_0 + \int_0^t f(s, x(s)) \, ds + \int_0^t \int k(t, \tau)k_1(\tau, x(\tau)) \, d\tau \, ds + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) \, d\tau \, ds, \quad t \in J \quad ........(6)
\]

By using the hypotheses \( H1, H3, H4 \), we have.

\[
\|Gx(t) - Gy(t)\|, \alpha \|Gx(t) - Gy(t)\|
\leq \left( \left\| \int_0^t f(s, x(s)) \, ds + \int_0^t \int k(t, \tau)k_1(\tau, x(\tau)) \, d\tau \, ds + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) \, d\tau \, ds \right\|, \alpha \right)

- \left( \left\| \int_0^t f(s, y(s)) \, ds + \int_0^t \int k(t, \tau)k_1(\tau, y(\tau)) \, d\tau \, ds + \int_0^t h(t, \tau)h_1(\tau, y(\tau)) \, d\tau \, ds \right\| \right)

\leq \left( \|f(s, x(s))\|, \alpha \|f(s, y(s))\| \right)

- \left( \left\| \int_0^t f(s, x(s)) \, ds + \int_0^t \int k(t, \tau)k_1(\tau, x(\tau)) \, d\tau \, ds + \int_0^t h(t, \tau)h_1(\tau, x(\tau)) \, d\tau \, ds \right\|, \alpha \right)
\[
\leq \left( \int_0^b \left( f(s, x(s)) - f(s, y(s)) \right) ds + \int_0^b \left[ \int_0^b \left( k(t, \tau) \left| \frac{\partial}{\partial \tau} (k, x(\tau)) - k, (\tau, y(\tau)) \right| d\tau \right) ds \right. \\
+ \left. \int_0^b h(t, \tau) \left| h, (\tau, x(\tau)) - h, (\tau, y(\tau)) \right| d\tau \right] ds, \alpha \right) \right) \left( \int_0^b \left( f(s, x(s)) - f(s, y(s)) \right) ds \right)
\]

By, x, y \in B

This implies that d(Fx, Fy) \leq \phi(d(x, y)) for every x, y \in B.

Now an application of Lemma 1 the operator has a unique point in B.

This means that the equation (1)–(2) has unique solution. This couples the proof of theorem (2).

**Application**

In this section we give example to illustrate the usefulness of our result. In equation (1), we define:
\[ k(t, s) = s, \quad k_1(s, x(s)) = \frac{x}{2}, \quad h(t, s) = s, \quad h_1(s, x(s)) = \frac{x^2}{2}, \quad f(s, x(s)) = \lambda x + s \]

\[ s, t \in [0, 1], \quad x \in C([0, 1], \mathbb{R}), \quad \lambda \text{ const} \tan t \]

And metric \( d(x, y) = (\|x - y\|_\alpha, \alpha \|x - y\|_\infty) \) on \( C([0, 1], \mathbb{R}) \) and \( \alpha \geq 0 \). Then clearly \( C([0, 1], \mathbb{R}) \) is complete cone metric space.

Now we have

\[
\begin{align*}
&\left( |k(t, s)[k_1(s, x(s)) - k_1(s, y(s))]|, \alpha |k(t, s)[k_1(s, x(s)) - k_1(s, y(s))]| \right) \\
&= \left( \left| s\left(\frac{x - y}{2}\right) \right|, \alpha \left| s\left(\frac{x - y}{2}\right) \right| \right) \\
&= \left( \frac{s}{2}|x - y|, \alpha \frac{s}{2}|x - y| \right) \\
&= \frac{s}{2}(|x - y|, \alpha |x - y|) \\
&\leq \frac{s}{2}(\|x - y\|_\infty, \alpha \|x - y\|_\infty) \\
&= M P_1^* \Phi^* (\|x - y\|_\infty, \alpha \|x - y\|_\infty),
\end{align*}
\]

Where \( P_1^* (s) = s \), which is continuous function of \([0,1] \times [0,1] \text{ int} \circ \mathbb{R}^+ \) and a comparison function \( \Phi^* : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \Phi^* (x, y) = (x, y) \). Similarly, we can show that
\([h(t,s)h_1(s,x(s)) - h_1(s,y(s)), \alpha h(t,s)h_1(s,x(s)) - h_1(s,y(s)))] \leq P_2^* \phi^* (\|x - y\|_e, \alpha \|x - y\|_e),\]

where \(P_2^* (s) = s\), which is continuous function of \([0,1] \times [0,1] \text{ int } R^+\). Similarly, we can show that

\[\|f(s,x(s)) - f(s,y(s))\| \alpha \|f(s,x(s)) - f(s,y(s))\| \leq \lambda \phi^* (\|x - y\|_e, \alpha \|x - y\|_e),\]

\(-1 < \lambda \leq 0\)

Moreover,

\[\int_0^1 [MP_1^* (s) + NP_2^* (s)] ds = \int_0^1 \frac{1}{2} s + \frac{1}{2} s \ ds = \frac{1}{2} < 1\]

Also

\[\int_0^1 [[MP_1^* (s) + MP_2^* (s)] ds = \int_0^1 \frac{1}{2} s + \frac{1}{2} s \ ds = \frac{1}{2} \ dt = \frac{1}{2} < 1\]

Also

\[\int_0^1 ds = 1\]

With these choices of function, all requirements of Theorem 1. And Theorem 2 are satisfied. Hence the existence and uniqueness are verified.

**References**


4) Biarca, Renatasatco, " Nonlinear Volterra integral equations in Henstock integrability setting" J.elec. diff.eqs.,


