Approximate Solution for Nonlinear System of Integro-Differential Equations of Volterra Type with Boundary Conditions

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Received 07 / 11 / 2013
Accepted 16 / 02 / 2014

ABSTRACT

In this study we investigate the approximation of the solution for nonlinear system of integro-differential equations of Volterra type with boundary conditions.

The numerical-analytic method of periodic solutions for ordinary differential equations of Samoilenko has been used of this work.

1. Introduction

The approximate periodic solutions for nonlinear systems of integro-differential equations have been used to study in many problems \[1,2,3,4,5\].

Ghada [2], used the method above to investigate the approximate periodic solution for nonlinear system of integro-differential equations of Volterra type which has the form:-
\[
\frac{dx(t)}{dt} = A(t)x(t) + \int_{0}^{t} K(t, s)F(t, s, x)ds + f(t)
\]

Also these investigations lend us to the improving and extending some work of Ghada [2].

Consider the following system of nonlinear integro-differential equation:
\[
\frac{dx(t)}{dt} = A(t)x(t) + \int_{0}^{t} K(t, s)F(t, s, x)ds + f(t), \quad \ldots \ldots (1.1)
\]

with boundary conditions
\[
Bx(0) + Cx(T) = d \quad \ldots \ldots (1.2)
\]

Here \( x \in G \subseteq R^n \), \( G \) is a closed and bounded domain subset of Euclidean spaces \( R^n \).

Let the vectors functions:
\[
f(t) = \left( f_1(t), f_2(t), \ldots, f_n(t) \right)
\]
\[
F(t, s, x) = \left( F_1(t, s, x), F_2(t, s, x), \ldots, F_n(t, s, x) \right),
\]
where the functions \( F(t, s, x) \) and \( f(t) \) are continuous, bounded on the domain:
\[
(t, s, x) \in [0, T] \times [0, T] \times G, \quad \ldots \ldots (1.3)
\]

where \( B = \left( B_{ij} \right) \), \( C = \left( C_{ij} \right) \) are constants positive matrices \( (n \times n) \).

Suppose that the functions \( F(t, s, x) \) and \( f(t) \) satisfies the following inequalities:
\[
\|F(t, s, x)\| \leq M, \quad \|f(t)\| \leq N \quad \ldots \ldots (1.4)
\]
\[
\|F(t, s, x_1) - F(t, s, x_2)\| \leq L\|x_1 - x_2\| \quad \ldots \ldots (1.5)
\]
for all \( t \in [0, T], \ s \in [0, T] \) and \( x, x_1, x_2 \in G \), where \( M, N \) and \( L \) are positive constants.

Let \( A(t), \ K(t, s) \) are \( (n \times n) \) non-negative matrices which is defined and continuous on (1.3), periodic in \( t \) of period \( T \), provided that:
\[
\|K(t, s)\| \leq H \quad \ldots \ldots (1.6)
\]
\[
\left\| \int_{0}^{\eta} e^{\eta A}d\eta \right\| \leq Q \quad \ldots \ldots (1.7)
\]

where \( -\infty < 0 \leq s \leq t \leq T < \infty \) and \( Q, H \) are a positive constants.

We define the non-empty sets as follows:
\[ G_f = G - \frac{T}{2} M_1 + \beta \quad \ldots \quad (1.8) \]

where \( M_1 = \frac{T}{2} \max_{t \in [0,T]} \bigg| f(t) \bigg| \) and \( \beta = \frac{T}{2} Q \left[ (C^{-1}A + E)x_0 - C^{-1}dQ^{-1} \right] \).

Furthermore, we suppose that:

\[ q = \left( QHLT \right) \left( \frac{T}{2} \right) < 1 \quad \ldots \quad (1.9) \]

By using lemma 3.1[5], we can state and prove the following lemma.

**Lemma 1.1**

Let \( f(t) \) and \( F(t,s,x) \) be continuous vector functions on the interval \([0,T]\) then the following:

\[
\left\| \frac{t}{T} \int_0^s K(s,\tau)F(s,\tau,x(\tau,x_0))d\tau + f(s) \right\| ds = \left\| -\frac{1}{T} \int_0^T \left( e^0_k - c^{-1} d^0 \right) ds \right\|
\]

\[
\left\| -\frac{1}{T} \int_0^T K(s,\tau)F(s,\tau,x(\tau,x_0))d\tau + f(s) \right\| ds \leq \alpha(t) M_1 + \beta
\]

Satisfying for \( 0 \leq t \leq T \) and \( \alpha(t) \leq \frac{T}{2} \) where \( \alpha(t) = 2T(1 - \frac{t}{T}) \), \( M_1 = \frac{T}{2} \max_{t \in [0,T]} \bigg| f(t) \bigg| \) and \( \beta = \frac{T}{2} Q \left[ (C^{-1}A + E)x_0 - C^{-1}dQ^{-1} \right] \).

**proof:**

\[
\left\| \frac{t}{T} \int_0^s K(s,\tau)F(s,\tau,x(\tau,x_0))d\tau + f(s) \right\| ds = \left\| -\frac{1}{T} \int_0^T \left( e^0_k - c^{-1} d^0 \right) ds \right\|
\]

\[
\left\| -\frac{1}{T} \int_0^T K(s,\tau)F(s,\tau,x(\tau,x_0))d\tau + f(s) \right\| ds \leq \alpha(t) M_1 + \beta
\]
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\[ \left\| 1 - \frac{t}{T} \right\| \left[ \int_0^t e^s \int_0^s K(s, \tau) F(s, \tau, x(\tau, x_0)) d\tau + f(s) \right] ds \right\| + \left\| \frac{t}{T} \int_0^t e^s \left[ \int_0^s \left( c^{-1} A + E \right) x_0 - c^{-1} d e^0 \right] ds \right\| + \left\| \frac{t}{T} \int_0^t e^s \left[ \int_0^s K(s, \tau) F(s, \tau, x(\tau, x_0)) d\tau + f(s) \right] ds \right\| \leq (1 - \frac{t}{T})[QHMT + QN] + \frac{t}{T} (T - t)[QHMT + QN] + \frac{t}{T} Q \left[ (c^{-1} A + E) x_0 - c^{-1} dQ^{-1} \right] = 2t(1 - \frac{t}{T})Q[QHMT + QN] + \frac{t}{T} Q \left[ (c^{-1} A + E) x_0 - c^{-1} dQ^{-1} \right] = \alpha(t) M_1 + \beta

2. Approximate Solution

The investigation of approximate solution of the problem (1.1) and (1.2) will be introduced by the following theorem:

**Theorem 1**

If the system (1.1) with boundary conditions (1.2) defined in the domain (1.3), continuous in \( t, x \) and satisfy the inequalities (1.4), (1.5) and (1.6), then the sequence of functions:

\[ x_{m+1}(t, x_0) = x_0 e^0 + \int_0^t e^0 \left[ \int_0^s K(s, \tau) F(s, \tau, x_m(\tau, x_0)) d\tau + f(s) \right] ds - \frac{1}{T} \left[ \int_0^t \left( c^{-1} A + E \right) x_0 - c^{-1} d e^0 \right] + \int_0^t \left[ \int_0^s K(s, \tau) F(s, \tau, x_m(\tau, x_0)) d\tau + f(s) \right] dt \right] ds \]

with

\[ x_0(t, x_0) = x_0 e^0 \]

periodic in \( t \) with period \( T \), converges uniformly when \( m \to \infty \) on the domain:

\[ (t, x_0) \in [0, T] \times G_f \]

… … (2.1)

… … (2.2)

to the limit function \( x(t, x_0) \) which is satisfying the integral equation:
\[
x(t, x_0) = x_0 e^0 + \int_0^t e^0 \left( \left[ \int K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] - \frac{1}{T} \left[ (c^{-1}A+E)x_0 - c^{-1}de^0 \right] + \frac{T}{t} \left[ \int_0^t K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] dt \right) ds
\]

its unique solution to (1.1) and satisfies the inequalities:

\[
\|x(t, x_0) - x_0\| \leq M_1 \frac{T}{2} + \beta \quad \cdots \cdots (2.3)
\]

\[
\|x(t, x_0) - x_m(t, x_0)\| \leq M_1 \left( \frac{T}{2} + \beta \right) \quad \cdots \cdots (2.5)
\]

for \( t \in [0, T] \), \( x_0 \in G_f \), \( m=0,1,2, \ldots \)

**Proof:**

Setting \( m=0 \) and using lemma 1.1 and the sequence of the functions (2.1) we get:

\[
\|x_1(t, x_0) - x_0\| = \left\| x_0 e^0 + \int_0^t e^0 \left( \left[ \int K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] - \frac{1}{T} \left[ (c^{-1}A+E)x_0 - c^{-1}de^0 \right] + \frac{T}{t} \left[ \int_0^t K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] dt \right) ds \right\|
\]

\[
\leq (1-\frac{t}{T})Q [QHMT + QN] + \frac{t}{T} (T-t) [QHMT + QN] + \frac{t}{T} Q \left[ (c^{-1}A+E)x_0 - c^{-1}dQ^{-1} \right]
\]

\[
= 2t(1-\frac{t}{T})Q [QHMT + N] + \frac{t}{T} Q \left[ (c^{-1}A+E)x_0 - c^{-1}dQ^{-1} \right]
\]

\[
= \alpha(t)M_1 + \beta
\]

\[
\|x_1(t, x_0) - x_0\| \leq \alpha(t)M_1 + \beta \leq M_1 \frac{T}{2} + \beta \quad \cdots \cdots (2.6)
\]

we get \( x_1(t, x_0) \in G_f \) for all \( t \in [0, T] \), \( x_0 \in G_f \).

By induction we have:

\[
\|x_n(t, x_0) - x_0\| \leq \left| 1-\frac{t}{T} \right| \left( \int_0^t e^0 \left[ \int K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] ds \right) + \frac{t}{T} \left( \int_0^t e^0 \left[ (c^{-1}A+E)x_0 - c^{-1}de^0 \right] ds \right) + \frac{t}{T} \left( \int_0^t e^0 \left[ \int K(s, \tau)F(s, \tau, x(\tau, x_0))d\tau + f(s) \right] ds \right)
\]
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\[ \leq 2t(1 - \frac{t}{T})Q[HMT + N] + \frac{t}{T} Q[c^{-1}A + E]x_0 - c^{-1}dQ^{-1} \]

\[ = \alpha(t)M_1 + \beta \]

\[ \|x_m(t, x_0) - x_0\| \leq \alpha(t)M_1 + \beta \leq M_1 \frac{T}{2} + \beta \] ...(2.7)

where \( x_m(t, x_0) \in G \), for all \( t \in [0,T] \), \( x_0 \in G_f \).

We prove now that the sequence (2.1) is uniformly convergent in (2.2). From (2.1), when \( m=1 \) we get:

\[ \|x_2(t, x_0) - x_1(t, x_0)\| = \left\| x_0 e^{0} + \int_{0}^{\tau} e^{0} \left( \int_{0}^{\tau} K(s, \tau) F(s, \tau, x_1(\tau, x_0)) d\tau + f(s) \right) ds - \frac{1}{T} \left( c^{-1}A + E \right) x_0 - c^{-1}d^0 - \int_{0}^{\tau} e^{0} \left( \int_{0}^{\tau} K(s, \tau) F(s, \tau, x_1(\tau, x_0)) d\tau + f(s) \right) dt \right\| ds - \right\| - x_0 e^{0} + \int_{0}^{\tau} e^{0} \left( \int_{0}^{\tau} K(s, \tau) F(s, \tau, x_1(\tau, x_0)) d\tau + f(s) \right) dt + f(s) \right) + \frac{1}{T} \left( c^{-1}A + E \right) x_0 - c^{-1}d^0 - \int_{0}^{\tau} e^{0} \left( \int_{0}^{\tau} K(s, \tau) F(s, \tau, x_1(\tau, x_0)) d\tau + f(s) \right) dt + f(s) \right) ds \]

\[ \leq (1 - \frac{t}{T}) \int_{0}^{\tau} Q[HLT(\alpha(t)M_1 + \beta)] ds + \frac{t}{T} \int_{0}^{\tau} Q[HLT(\alpha(t)M_1 + \beta)] ds \]

\[ \leq \frac{T}{2} (QHLT)(\alpha(t)M_1 + \beta) \]

\[ = \Lambda(\alpha(t)M_1 + \beta) \]

therefore

\[ \|x_2(t, x_0) - x_1(t, x_0)\| \leq \Lambda \left( M_1 \frac{T}{2} + \beta \right) \]

Now when \( m=2 \) we get the following:

\[ \|x_3(t, x_0) - x_2(t, x_0)\| \leq (1 - \frac{t}{T}) \int_{0}^{\tau} Q \int_{0}^{\tau} \|x_2(\tau, x_0) - x_1(\tau, x_0)\| d\tau \] ds +

\[ + \frac{t}{T} \int_{0}^{\tau} Q \int_{0}^{\tau} \|x_2(\tau, x_0) - x_1(\tau, x_0)\| d\tau \] ds

\[ \leq \frac{T}{2} (QHLT) \Lambda \left( M_1 \frac{T}{2} + \beta \right) \]
\[ \|x_3(t, x_0) - x_2(t, x_0)\| \leq \Lambda^2 \left( M_1 \frac{T}{2} + \beta \right). \]

By mathematical induction we have:
\[ \|x_{m+1}(t, x_0) - x_m(t, x_0)\| \leq \Lambda^m \left( M_1 \frac{T}{2} + \beta \right) \]
for \( m=0,1,2,\ldots \).

By using the condition (1.9), we have
\[ \lim_{m \to \infty} \Lambda^m = 0 \]
… … (2.9)

So that the right hand from (2.8) equal zero when \( m \to \infty \). Suppose that \( \varepsilon > 0 \), we get a positive integer \( n \) such that \( n < m \), and satisfied the next estimation for all \( m \):
\[ \|x_{m+p}(t, x_0) - x_m(t, x_0)\| < \varepsilon, \text{ for } P \in \mathbb{N}. \]

Then according to the definition of uniformly convergent, we find that the sequence \( \{x_m(t, x_0)\}_{m=0}^{\infty} \) is uniformly convergent from the function \( x(t, x_0) \) and this function be continuous on the same interval.

Putting
\[ \lim_{m \to \infty} x_m(t, x_0) = x(t, x_0) \]
… … (2.10)

Since the sequence of functions \( x_m(t, x_0) \) is continuous on the domain (2.2) then the limiting function \( x(t, x_0) \) is also continues on the same domain.

Also by using lemma1.1 and the relation (2.10), then the inequalities (2.4) and (2.5) are satisfies for all \( m \).

Finally, we show that \( x(t, x_0) \) is unique solution of the problem (1.1) and (1.2). On country we suppose that there is at least one different solution \( \hat{x}(t, x_0) \) of the problem (1.1) and (1.2), then:
\[
\hat{x}(t, x_0) = x_0 e^0 + \int_0^t e^\eta \left( \int_0^\tau K(s, \tau) F(s, \tau, \hat{x}(\tau, x_0))d\tau + f(s) \right) d\tau.
\]

… … (2.11)
Now we prove that \( \hat{x}(t, x_0) = x(t, x_0) \) for \( x_0 \in D_f \), by proving the following inequality:

\[
\|\hat{x}(t, x_0) - x_m(t, x_0)\| \leq \Lambda^m \left( M^*_1 \frac{T}{2} + \beta \right)
\]

... (2.12)

where \( M^*_1 = Q[HRT + N] \), \( R = \max_{t \in [0, T]} \| F(s, t, \hat{x}) \| \).

let \( m=0 \) in (2.1) and from (2.11) we find:

\[
\|\hat{x}(t, x_0) - x_0\| = \left\| x_0 e^0 + \int_0^t e^0 \left( \int_0^t (K(s, \tau)F(s, \tau, \hat{x}(\tau, x_0))d\tau + f(s) \right) ds \right\|
\]

\[
\leq \left(1 - \frac{T}{t}\right) \int_0^t e^0 \left( \int_0^t (K(s, \tau)F(s, \tau, \hat{x}(\tau, x_0))d\tau + f(s) \right) ds + \frac{T}{t} \int_0^t e^0 \left[ (c^{-1} A + E) x_0 - c^{-1} dQ^{-1} \right] ds
\]

\[
= \alpha(t) M^*_1 + \beta
\]

\[
\|\hat{x}(t, x_0) - x_0\| \leq \alpha(t) M^*_1 + \beta \leq M^*_1 \frac{T}{2} + \beta
\]

and when \( m=1 \) in (2.1) and from (2.11) we find:

\[
\|\hat{x}(t, x_0) - x_1(t, x_0)\| \leq \left(1 - \frac{T}{t}\right) \int_0^t e^0 \left( \int_0^t (K(s, \tau)(F(s, \tau, \hat{x}(\tau, x_0)) - F(s, \tau, x_0(\tau, x_0))d\tau \right) ds +
\]

\[
+ \frac{T}{t} \int_0^t e^0 \left( \int_0^t (K(s, \tau)(F(s, \tau, \hat{x}(\tau, x_0)) - F(s, \tau, x_0(\tau, x_0))d\tau \right) ds
\]

\[
\leq (1 - \frac{T}{t}) \int_0^t Q[HLT(\alpha(t) M^*_1 + \beta)] ds + \frac{T}{t} \int_0^t Q[HLT(\alpha(t) M^*_1 + \beta)] ds
\]
\[
\leq \frac{T}{2}(QHLT)(\alpha(t)M_1^* + \beta)
\]
\[
= \Lambda(\alpha(t)M_1^* + \beta)
\]
\[
\|\hat{x}(t, x_0) - x_1(t, x_0)\| \leq \Lambda\left( M_1^* \frac{T}{2} + \beta \right)
\]

and when \( m=2 \) in (2.1) and from (2.11) we find:
\[
\|\hat{x}(t, x_0) - x_2(t, x_0)\| = \left\| x_0 e^0_+ + \int_0^t e^0_+ \left[ \int_0^s K(s, \tau)F(s, \tau, \hat{x}(\tau, x_0))d\tau + f(s) \right]d\tau - \right.
\]
\[
\left. - \frac{1}{T} \left[ (c^{-1}A+E)x_0 - c^{-1}de^0_+ \right] + \int_0^t \left[ \int_0^s K(s, \tau)F(s, \tau, \hat{x}(\tau, x_0))d\tau + f(s) \right]dt \right\| ds -
\]
\[
\left. - x_0 e^0_+ + \int_0^t e^0_+ \left[ \int_0^s K(s, \tau)F(s, \tau, x_1(\tau, x_0))d\tau + f(s) \right]d\tau \right.
\]
\[
\left. + \frac{1}{T} \left[ (c^{-1}A+E)x_0 - c^{-1}de^0_+ \right] + \int_0^t \left[ \int_0^s K(s, \tau)F(s, \tau, x_1(\tau, x_0))d\tau + f(s) \right]dt \right\| ds
\]
\[
\leq (1 - \frac{t}{T}) \int_0^T HLT\Lambda\left( M_1^* \frac{T}{2} + \beta \right) ds + \frac{t}{T} \int_0^T HLT\Lambda\left( M_1^* \frac{T}{2} + \beta \right) ds
\]
\[
\leq \frac{T}{2}(QHLT)\Lambda\left( M_1^* \frac{T}{2} + \beta \right)
\]
\[
\|\hat{x}(t, x_0) - x_2(t, x_0)\| \leq \Lambda^2\left( M_1^* \frac{T}{2} + \beta \right)
\]

we find that the inequality (2.12) is satisfying when \( m=0,1,2 \).

Suppose that the inequality (2.12) is satisfying when \( m=p \) as the following inequality:
\[
\|\hat{x}(t, x_0) - x_p(t, x_0)\| \leq \Lambda^p\left( M_1^* \frac{T}{2} + \beta \right)
\]
\[ \ldots \ldots \text{(2.13)} \]

Next we will proof the following inequality:
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\[
\left\| \hat{x}(t, x_0) - x_{p+1}(t, x_0) \right\| \leq \Lambda^{p+1}\left( M_1^* \frac{T}{2} + \beta \right) \quad \ldots \ldots (2.14)
\]

Now

\[
\left\| \hat{x}(t, x_0) - x_{p+1}(t, x_0) \right\| = \left\| x_0 e^0 + \int_0^T e^0 \left( \int_0^{s} K(s, \tau) F(s, \tau, \hat{x}(\tau, x_0)) d\tau + f(s) \right) d\tau \right\| - \frac{1}{T} \left[ \left( c^{-1} A + E \right) x_0 - c^{-1} de^0 \right] + \int_0^T \int_0^{s} K(s, \tau) F(s, \tau, \hat{x}(\tau, x_0)) d\tau + f(s) dt \right) ds - \left[x_0 e^0 + \int_0^T e^0 \left( \int_0^{s} K(s, \tau) F(s, \tau, x_p(\tau, x_0)) d\tau + f(s) \right) d\tau \right] + \frac{1}{T} \left[ \left( c^{-1} A + E \right) x_0 - c^{-1} de^0 \right] + \int_0^T \int_0^{s} K(s, \tau) F(s, \tau, x_p(\tau, x_0)) d\tau + f(s) dt \right) ds
\]

\[
\leq \frac{T}{2} (QHLT) \Lambda^p \left( M_1^* \frac{T}{2} + \beta \right)
\]

then

\[
\left\| \hat{x}(t, x_0) - x_{p+1}(t, x_0) \right\| \leq \Lambda^{p+1}\left( M_1^* \frac{T}{2} + \beta \right)
\]

Thus we find that the inequality (2.15) is satisfying when \( m=0,1,2,\ldots \).

From the conditions (1.9), (2.10) we get:
\[
\hat{x}(t, x_0) = \text{Lim}_{m \to \infty} x_m(t, x_0) = x(t, x_0).
\]

3. Existence of solution

The problem of existence solution of the problem (1.1), (1.2) is uniquely connected with the existence of zeros of the function \( \Delta = \Delta(x_0) \) which has the form:

\[
\Delta(x_0) = \frac{1}{T} e^0 \left[ c^{-1} A + E \right] x_0 - c^{-1} de^0 + \int_0^T \int_0^{s} K(s, \tau) F(s, \tau, x(\tau, x_0)) d\tau + f(s) dt \right]
\]

\[
\ldots \ldots (3.1)
\]
Since this functions are approximately determined from the sequence of functions:
\[ \Delta_m(x_0) = \frac{1}{T} \int_{QHLT} \left[ \left( e^{A+\varepsilon} \right) x_0 - e^{-\frac{t}{T}} \int_0^t K(s,\tau) F(s,\tau,\sigma(x_0)) d\tau + f(s) \right] d\varepsilon \]
for \( m=0,1,2,\ldots \).

**Theorem 2**

Let all assumptions and conditions of theorem 1 be given, then the following inequality
\[ \left\| \Delta(x_0) - \Delta_m(x_0) \right\| \leq \Lambda^{m+1} \left( M_1 + \frac{2}{T} \beta \right) \]
for all \( m \geq 0 \) and \( x_0 \in D_f \).

**Proof:**

By (3.1) and (3.2) we get:
\[ \left\| \Delta(x_0) - \Delta_m(x_0) \right\| = \frac{1}{T} \int_{QHLT} \left[ \left( e^{A+\varepsilon} \right) x_0 - e^{-\frac{t}{T}} \int_0^t K(s,\tau) F(s,\tau,\sigma(x_0)) d\tau + f(s) \right] d\varepsilon \]
\[ \leq \frac{1}{T} \int_{QHLT} \left[ \left( e^{A+\varepsilon} \right) x_0 - e^{-\frac{t}{T}} \int_0^t K(s,\tau) F(s,\tau,\sigma(x_0)) d\tau + f(s) \right] d\varepsilon \]
\[ \leq \frac{1}{T} \int_{QHLT} \left[ \int_0^t \left\| F(s,\tau,\sigma(x_0)) - F(s,\tau,\sigma(x_0)) \right\| d\tau \right] d\varepsilon \]
\[ \leq \frac{1}{T} \int_{QHLT} \left[ \Lambda^{m+1} \left( M_1 + \frac{2}{T} \beta \right) \right] d\varepsilon \]

By (2.5) we find
\[ \leq \frac{1}{T} \int_{QHLT} \left[ \Lambda^{m+1} \left( M_1 + \frac{2}{T} \beta \right) \right] d\varepsilon \]
\[ = \Lambda^{m+1} \left( M_1 + \frac{2}{T} \beta \right) \]

then
\[ \left\| \Delta(x_0) - \Delta_m(x_0) \right\| \leq \Lambda^{m+1} \left( M_1 + \frac{2}{T} \beta \right) \]
for all \( m=0,1,2,\ldots \).
Theorem 3
If the function $\Delta(x_0)$ is defined by:

$$\Delta(x_0) = \frac{1}{T} e^s \left[ \left( c^{-1} A + E \right)x_0 - c^{-1} d e^s + \int_0^T \left[ K(s, \tau) F(s, \tau, x(\tau, x_0)) d\tau + f(s) \right] dt \right]$$

where the function $x(t, x_0)$ is limit of function (2.1) then the inequalities:

$$\|\Delta(x_0)\| \leq M_1 + \frac{\beta}{T}$$

where $M_1 = Q[HMT + N]$, $\beta = \frac{T}{T} Q[\left( c^{-1} A + E \right)x_0 - c^{-1} d Q^{-1}]$.

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| \leq \left[ \left( c^{-1} A + E \right) + \frac{2}{T} \Lambda A c^{-1} \right] \frac{1}{T} \|x_0^1 - x_0^2\| Q$$

for $x_0, x_0^1, x_0^2 \in D_f$.

Proof:
From the continuity of the function $\Delta(x_0)$, then

$$\|\Delta(x_0)\| = \frac{1}{T} e^s \left[ \left( c^{-1} A + E \right)x_0 - c^{-1} d e^s + \int_0^T \left[ K(s, \tau) F(s, \tau, x(\tau, x_0)) d\tau + f(s) \right] dt \right]$$

$$\leq \frac{1}{T} Q[\left( c^{-1} A + E \right)x_0 - c^{-1} d Q^{-1}] + \frac{1}{T} Q \left[ \int_0^T \left[ H M d\tau + N \right] dt \right]$$

$$\leq \beta + T \frac{1}{T} Q[HMT + N] dt$$

$$= \beta + M_1$$

$$\|\Delta(x_0)\| \leq M_1 + \frac{\beta}{T}.$$

Now from (3.4) we get:

$$\|\Delta(x_0^1) - \Delta(x_0^2)\| = \frac{1}{T} e^s \left[ \left( c^{-1} A + E \right)x_0^1 - c^{-1} d e^s + \int_0^T \left[ K(s, \tau) F(s, \tau, x(\tau, x_0^1)) d\tau + f(s) \right] dt \right]$$

$$- \frac{1}{T} e^s \left[ \left( c^{-1} A + E \right)x_0^2 - c^{-1} d e^s + \int_0^T \left[ K(s, \tau) F(s, \tau, x(\tau, x_0^2)) d\tau + f(s) \right] dt \right]$$
\[ \| \Delta(x^1_0) - \Delta(x^2_0) \| \leq \frac{1}{T} Q(c^{-1}A + E)\|x^1_0 - x^2_0\| + \frac{2}{T} QHLT \frac{T}{2} \|x(t,x^1_0) - x(t,x^2_0)\| \]

then

\[ \| \Delta(x^1_0) - \Delta(x^2_0) \| \leq \frac{1}{T} Q(c^{-1}A + E)\|x^1_0 - x^2_0\| + \frac{2}{T} \Lambda \|x(t,x^1_0) - x(t,x^2_0)\| \]

\[ \cdots \cdots (3.7) \]

Since the functions \( x(t,x^1_0) \), \( x(t,x^2_0) \) are the solution of integral equation:

\[ x(t,x^\mu_0) = x^\mu_0 e^0 + \int_0^t e^0 \left( \int_0^s K(s,\tau)F(s,\tau,x(x^\mu_0))d\tau + f(s) \right) ds + \int_0^t \left( -c^{-1}A + E \right)x^\mu_0 e^0 + \int_0^s \left( \frac{c^{-1}A}{T} \right)K(s,\tau)F(s,\tau,x(x^\mu_0))d\tau + f(s) \right) ds \]

\[ \cdots \cdots (3.8) \]

where \( \mu = 1,2 \).

Then by (3.8) and lemma 1.1, we get:

\[ \| x(t,x^1_0) - x(t,x^2_0) \| = \left| x^1_0 e^0 + \int_0^t e^0 \left( \int_0^s K(s,\tau)F(s,\tau,x(x^1_0))d\tau + f(s) \right) ds + \int_0^t \left( -c^{-1}A + E \right)x^1_0 e^0 + \int_0^s \left( \frac{c^{-1}A}{T} \right)K(s,\tau)F(s,\tau,x(x^1_0))d\tau + f(s) \right) ds \]

\[ \cdots \cdots (3.9) \]
\[
\leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q + \left( 1 - \frac{t}{T} \right) \int_0^t \int_0^s \left[ K(s, \tau) \left( F(s, \tau, x(\tau, x^1_0)) - F(s, \tau, x(\tau, x^2_0)) \right) \right] d\tau \ ds + \\
+ \left( 1 - \frac{t}{T} \right) Q \int_0^t \int_0^s \left[ K(s, \tau) \left( F(s, \tau, x(\tau, x^1_0)) - F(s, \tau, x(\tau, x^2_0)) \right) \right] d\tau \ ds
\]

\[
\leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q + \left( 1 - \frac{t}{T} \right) t(QHLT) \left\| x(t, x^1_0) - x(t, x^2_0) \right\| + \\
+ \left( 1 - \frac{t}{T} \right)(QHLT) \left\| x(t, x^1_0) - x(t, x^2_0) \right\|
\]

\[
\leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q + \frac{T}{2} (QHLT) \left\| x(t, x^1_0) - x(t, x^2_0) \right\|
\]

then

\[
\left\| x(t, x^1_0) - x(t, x^2_0) \right\| \leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q + \Lambda \left\| x(t, x^1_0) - x(t, x^2_0) \right\|
\]

\[
\left\| x(t, x^1_0) - x(t, x^2_0) \right\| - \Lambda \left\| x(t, x^1_0) - x(t, x^2_0) \right\| \leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q
\]

\[
(1 - \Lambda) \left\| x(t, x^1_0) - x(t, x^2_0) \right\| \leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q
\]

\[
\left\| x(t, x^1_0) - x(t, x^2_0) \right\| \leq \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q
\]

Substituting (3.9) in (3.7) we get (3.6):

\[
\left\| \Delta(x^1_0) - \Delta(x^2_0) \right\| \leq \frac{1}{T} Q(c^{-1} A + E) \left\| x^1_0 - x^2_0 \right\| + \frac{2}{T} \Lambda \left\| x(t, x^1_0) - x(t, x^2_0) \right\|
\]

\[
\left\| \Delta(x^1_0) - \Delta(x^2_0) \right\| \leq \frac{1}{T} Q(c^{-1} A + E) \left\| x^1_0 - x^2_0 \right\| + \frac{2}{T} \Lambda \frac{A}{Tc} \left\| x^1_0 - x^2_0 \right\| Q
\]

\[
\left\| \Delta(x^1_0) - \Delta(x^2_0) \right\| \leq \left( c^{-1} A + E \right) + \frac{2}{T} \Lambda c^{-1} \left[ \frac{1}{T} \right] \left\| x^1_0 - x^2_0 \right\| Q
\]
Remark 2.1[4].

The theorem 3 ensures the stability solution of the system (1.1), when there is a slight change in the point $x_0$ accompanied with a noticeable change in the function $\Delta = \Delta(t, x_0)$.

REFERENCES


