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$h\alpha$-Open Sets in Topological Spaces

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Abstract
In our work a new type of open sets is introduced and defined as follows: If for each set that is not empty M in X, $M \neq X$ and $M \in \tau^{\propto}$such that $A \subseteq int(A \cup M)$, then A in (X,τ) is named $h \propto$-open set. We also go through the relationship between $h \propto$- open sets and a variety of other open set types as h-open sets, open sets, semi-open sets and $\propto$-open sets. We proved that each h-open and open set is $h \propto$-open and there is no relationship between $\propto$-open sets and semi-open sets with $h \propto$-open sets. Furthermore, we begin by introducing the concepts of $h \propto$-continuous mappings, $h \propto$-open mappings, $h \propto$-irresolute mappings, and $h \propto$-totally continuous mappings, we proved that each h-continuous mapping in any topological space is $h \propto$-continuous mapping, each continuous mapping in any topological space is $h \propto$-continuous mapping and there is no relationship between $\propto$-continuous mappings and semi-continuous mappings with $h \propto$-continuous mappings as well as some of its features. Finally, we look at some of the new class's separation axioms.

Keywords: $h \propto$-open set, $h \propto$-continuous mapping, $h \propto$-open mapping, $h \propto$-totally continuous mapping, $h \propto$-irresolute mapping.

المجاميع المفتوحة من النمط$-h\alpha$ في الفضاءات التبولوجية

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الخلاصة:
في هذا البحث قدمنا صنفا جديدا من المجاميع المفتوحة والذي عرفناه بالشكل الاتي: لكل مجموعة مفتوحة غير خالية M في X ، $M \neq X$ و $M \in \tau^{\propto}$ ، بحيث ان $A \subseteq int(A \cup M)$ ، عندئذ يقال للمجموعة A بانها مفتوحة من النمط$-h \propto$. ايضا اعطينا العلاقات بين المجاميع المفتوحة من النمط$-h \propto$ وعدة اصناف اخرى متنوعة من المجاميع مثل المجاميع المفتوحة من النمط-h، المجاميع شبه المفتوحة، المجاميع المفتوحة والمجاميع المفتوحة من النمط-∝، حيث برهننا بان كل مجموعة مفتوحة من النمط-h في أي فضاء تبولوجي تكون مفتوحة من النمط$-h \propto$ وكل مجموعة مفتوحة تكون مفتوحة من النمط$-h \propto$ وانه لا توجد علاقة بين المجاميع شبه المفتوحة والمجاميع المفتوحة من النمط-∝ مع المجاميع المفتوحة من النمط$-h \propto$ فضلا عن ذلك قدمنا تعاريف الدوال المستمرة والدوال المفتوحة والدوال المترددة والدوال المستمرة التامة من النمط$-h \propto$ مع اعطاء بعض خصائصها، حيث برهننا بان كل تطبيق مستمر من النمط-h في أي فضاء تبولوجي يكون مستمرا من النمط$-h \propto$ وكل تطبيق مستمر يكون مستمر من النمط$-h \propto$ وانه لا توجد علاقة بين التطبيقات شبه المستمرة والتطبيقات المستمرة من النمط-∝ مع التطبيقات المستمرة من النمط$-h \propto$. اضافة الى اعطاء اصناف من بديهيات الفصل الخاصة بهذا الصنف من المجاميع المفتوحة.

الكلمات المفتاحية: المجموعة المفتوحة من النمط $h\propto$ ، التطبيقات المستمرة من النمط $h\propto$ ، التطبيقات المفتوحة من النمط $h\propto$ ، التطبيقات المستمرة التامة من النمط $h\propto$ ، التطبيقات المترددة من النمط $h\propto$.

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1. Introduction and preliminaries

Njasted [6] introduced α -open set, Abbas [3] presented h-open set, h-irresolute mapping, and h-homeomorphism, and Levine [4] defined the semi-open set and semi-continuous function. α – continuous and α -open mappings were introduced by Mashhour, Hasanein, and EL-Deeb [5], totally continuous functions were introduced by Noiri [7], and irresolute functions were introduced by Crossley [2]. The fundamental aim of the work is to present and examine a new concepts of open sets known as "$h\alpha$ -open sets", as well as to look at some of the connections between open sets, semi-open sets, α -open sets, and "h-open sets". The concepts of "$h\alpha$—continuous", "$h\alpha$--open mapping", and "$h\alpha$--irresolute mapping" are introduced. We also look into some of the aspects of these mappings in section 2. In section 3, we look into the relationships among "$h\alpha$--continuous mapping" and various types of "continuous mappings", $h\alpha$--open mapping and various types of open mappings, and $h\alpha$--irresolute mapping and various types of "irresolute mappings". We also make a comparison between "$h\alpha$—homeomorphism" and "h-homeomorphism". Section 4 introduces a new class of mappings known as "$h\alpha$ -totally continuous mappings" and examines some of their fundamental properties. Finally, we look at some of the new class's separation axioms, especially, $T_{0h\alpha}$ and $T_{1h\alpha}$. We denoted the topological spaces (X,τ) and (Y,σ) simply by X and Y, respectively. Open sets (resp. closed sets) by(os), (cs), topological spaces by TS . $cl(A)$ (resp. $int(A)$) denotes "the closure" (resp. interior) of a subset A of X.

Definition 1.1 It is defined that a subset A of a topological space X is referred to as a:

1. "Semi-open set" denoted by(s - os) if $\exists\ U \in \tau$ as a result $U \subseteq A \subseteq cl\ (U)$[4]
2. "$\propto$-open set" denoted by (α - os), if $A \subseteq int(cl(\ int(A)))$ [6]
3. "h-open set" denoted by (h - os), if for each set that is not empty U in X, $U \neq X$ and $U \in \tau$, as a result $A \subseteq int(A \cup U)$ [3]
4. If A is both open and closed, it is said to be "clopen set" denoted by (cl - os).

The family of all ($s-os$) (resp.($\alpha-os$), ($h-os$)) sets of TS is denoted by $\tau^s(resp.\tau^\alpha,\tau^h)$. The complement of ($s-os$) (resp.($\alpha-os$), ($h-os$)) sets of TS X is called "semi-closed" ($s-cs$) (resp. "$\propto$-closed" ($\alpha-cs$), "h-closed" ($h-cs$)) sets.

Definition1.2. Assume that X and Y are TS , a mapping $f : X \to Y$ is named:

1. "Semi-continuous" denoted by ($scontm$) [4] if $f^{-1}(G)$ is (s-os) in X, $\forall\ G\ \in$(os) in Y.
2. "$\propto$-continuous" denoted by ($\alpha-contm$)[5] if $f^{-1}(G)$ is ($\propto$-os) in X, $\forall\ G\ \in$(os) in Y.
3. "h-continuous" denoted by ($h-contm$)[3] if $f^{-1}(G)$ is (h-os) in X, $\forall\ G\ \in$(os) in Y.
4. "Totally-continuous" denoted by ($tconm$)[7] if $f^{-1}(G)$ is (cl-os) in X, $\forall\ G\ \in$(os) in Y.
5. "Irresolute" denoted by ($irem$)[2] if $f^{-1}(G)$ is (s-os) in X, $\forall\ G\ \in$(s-os) in Y.
6. "h-irresolute" denoted by ($h-irem$) [3] if $f^{-1}(G)$ is (h-os) in X, $\forall\ G\ \in$(h-os) in Y.
7. "Semi-open" denoted by ($s-om$)[1] if $f(G)$ is (s-os) in Y, $\forall\ G\ \in$(os) in X.
8. "$\propto$-open" denoted by ($\alpha-om$) [5] if $f(G)$ is ($\propto$-os) in Y, $\forall\ G\ \in$(os) in X.
9. "h-open" denoted by ($h-om$) [3] if $f(G)$ is (h-os) in Y, $\forall\ G\ \in$(os) in X.

Definition1.3. Let's say X and Y are TS, a "bijective mapping" $f : X \to Y$ is named "h-homeomorphism" denoted by ($h-homo$) [3] if f is "h-continuous" and "h-open".

Lemma1.4. Each (os) in TS is ($h-os$) *[3].*

Lemma1.5. Each (os) in TS is ($\alpha-os$) *[6].*

2. $h\propto$-Open Sets in Topological Spaces.

Definition2.1. A subset A of TS X is named "$h \propto$ - open set" denoted by $(h\alpha - os)$ if for each set that is not empty U in $X, U \neq X$ and $U \in \tau^{\propto}$, as a result $A \subseteq int(A \cup U)$. The opposite of the "$h \propto$-open set" is named "$h \propto$-closed set" denoted by $(h\alpha - cs)$, we denoted the collection of all $(h\alpha - os)$ of TS X by $\tau^{h\alpha}$.
Example2.2. If $X = \{2,4,6\}$ and $\tau = \{\emptyset, X, \{4\}, \{4,6\}\}$, $\tau^{h\propto} = \{\emptyset, X, \{4\}, \{6\}, \{4,6\}, \{2,6\}\}$
Lemma2.3. Each $(h - os)$ in any TS is $(h\alpha - os)$.
Proof: Let X be TS and $A \subseteq X$ be any $(h - os)$. Henceforth,, for each set that is not empty U in X, $U \neq X$ and $U \in \tau$, within "$A \subseteq int(A \cup U)$". Since, each os is $(\alpha - os)$[6][5], then, U is $(\alpha - os)$. Thus, A is $(h\alpha - os)$.
Lemma2.4. Any (os) in any TS is $(h\alpha - os)$.
Proof: Let X be any TS, $A \subseteq X$ be any (os). Since each os is $(h - os)$[3][2], then A is $(h - os)$. By lemma (2.3), we get A is $(h\alpha - os)$. ■
Example2.5. Let $X = \{1,3,5\}$ and $\tau = \{\emptyset, X, \{1\}, \{1,3\} \{1,5\}\}$
$\tau^{h\propto} = \{\emptyset, X, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}\}$. Then $\{3\}$ is $(h\alpha - os)$ but it is not (os).
Remark2.6. There is no relationship between $(\alpha - os)$ and $(s - os)$ with $(h\alpha - os)$ as seen in the examples below:
Example2.7. Let $X = \{2,4,6\}$. Now,
1- Let $\tau = \{\emptyset, X, \{4\}, \{4,6\}\}$. then, $\{2,6\}$ is $(h\alpha - os)$ but it is not $(\alpha - os)$ and it is not $(s - os)$.
2- Let $\tau = \{\emptyset, X, \{6\}\}$. then, $\{4,6\}$ is $(\alpha - os)$ and $(s - os)$ but it is not $(h\alpha - os)$.

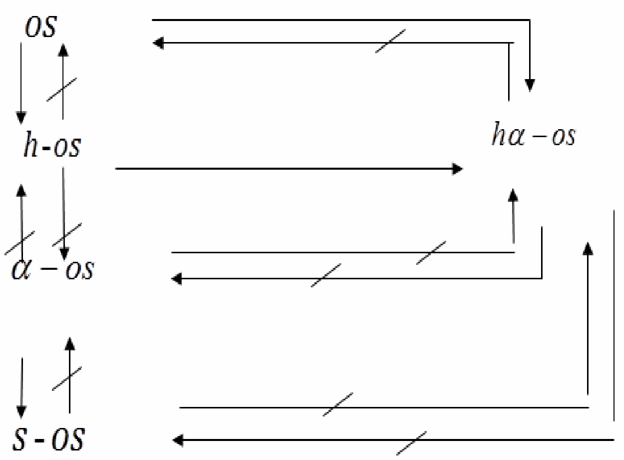

Fig1. The relationships of $h\alpha$-open sets with other classes mentioned above

3. $h \propto$-Continuous Mappings and $h \propto$-Homeomorphisms.
Definition3.1. A mapping "$f : X \to Y$" is named "$h \propto$-continuous" denoted by $(h\alpha - conm)$, if $f^{-1}(G)$ is $(h\propto\text{-}os)$ in X, $\forall$ $G \in (os)$ in Y.
Example3.2. Let $X = Y = \{1,3,5\}$ and $\tau = \{\emptyset, X, \{3\}, \{3,5\}\}$ then
$\tau^{h\propto} = \{\emptyset, X, \{3\}, \{5\}, \{3,5\}, \{1,5\}\}$, $\sigma = \{\emptyset, Y, \{3,5\}\}$. The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha - conm)$.
Proposition3.3. *Each $(h - conm)$ is $(h\alpha - conm)$.*

Proof: Suppose that $f : X \to Y$ be $(h-conm)$ and V is any $(os)Y$. because such f is $(h-conm)$ then $f^{-1}(V)$ is $(h-os)$ in X. because, each $(h-os)$ is $(h\alpha-os)$ by lemma (2.3), then, $f^{-1}(V)$ is $(h\alpha-os)$ in X. Henceforth f is $(h\alpha-conm)$. ■

Proposition3.4. Each $(conm)$ is $(h\alpha-conm)$.

Proof: Suppose that " $f : X \to Y$ " be $conm$ and V is any (os) in Y. Because such f is $(conm)$ then $f^{-1}(V)$ is os in X. Since; each (os) is $(h\alpha-os)$ by lemma (2.4), then, $f^{-1}(V)$ is $(h\alpha-os)$ in X. Henceforth, f is $(h\alpha-conm)$. ■

Remark3.5. The following example demonstrates that $(h\alpha-conm)$ is not required $(s-conm)$ and $(\alpha-conm)$.

Example3.6. Let $X = Y = \{2,4,6\}$ and $\tau = \{\emptyset, X, \{4\}, \{4,6\}\}$
$\sigma = \{\emptyset, Y, \{2,6\}\}$, $\tau^{h\alpha} = \{\emptyset, X, \{4\}, \{6\}, \{2,6\}, \{4,6\}\}$. The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha-conm)$, but f isn't $(s-conm)$ and isn't $(\alpha-conm)$ because for (os) $\{2,6\}$, $f^{-1}(\{2,6\}) = \{2,6\} \notin \tau^s = \tau^\alpha$.

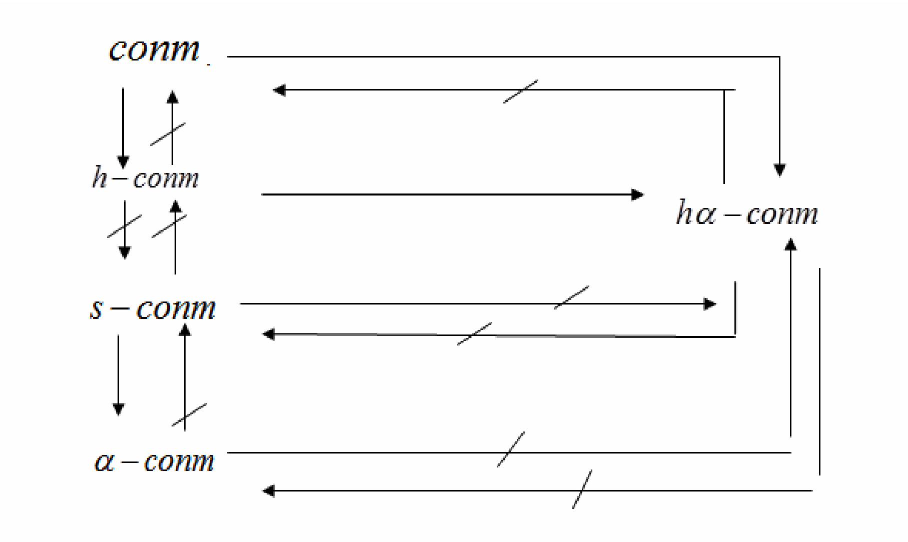

Fig2. The relationships of $h\alpha-$ continuous mappings with other continuous mappings

Definition3.7. A mapping $f : X \to Y$ is named" h ∝-open" denoted by $(h\alpha-om)$, if the image of each (os) in X is $(h\alpha-os)$ in Y.

Example3.8. Let $X = Y = \{4,6,8\}$ and $\tau = \{\emptyset, X, \{4,6\}\}$, $\sigma = \{\emptyset, Y, \{8\}\}$, $\tau^{h\alpha} = \{\emptyset, Y, \{8\}, \{4,6\}\}$. The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha-om)$.

Proposition3.9. Each $(h-om)$ is $(h\alpha-om)$.

Proof: Assume that $f : X \to Y$ is $(h-om)$ and V is any (os) in X. Then since, f is $(h-om)$, we get $f(V)$ is $(h-os)$in Y. Since, each $(h-os)$ is $(h\alpha-os)$by "lemma (2.3)", we get, $f(V)$ is $(h\alpha-os)$ in Y. Henceforth, f is $(h\alpha-om)$. ■

Proposition3.10. *Any* (om) *is* $(h\alpha-om)$.

Proof: Assume that $f : X \to Y$ is (om) and V is (os) in X. Since, then, $f(V)$ is (os) in Y. Since, each (os) is $(h\alpha-os)$ by lemma (2.4), then, $f(V)$ is $(h\alpha-os)$ in Y. Henceforth, f is $(h\alpha-om)$. ■

Remark3.11. "The following example shows that $(h\alpha-om)$ need not be$(s-om)$ and $(\alpha-om)$"

Example3.12. Let $X = Y = \{1,2,3\}$ and $\tau = \{\emptyset, X, \{2,3\}\}$, $\sigma = \{\emptyset, Y, \{1\}\}$, $\tau^{h\alpha} = \{\emptyset, Y, \{1\}, \{2,3\}\}$.

" $f : X \to Y$ " is defined by $f(1) = 1, f(2) = 3, f(3) = 2$. Clearly, f is $(h\alpha-om)$, but f is not $(s-om)$ and *it* is not $(\alpha-om)$ because for (os) $\{2,3\}$, $f(\{2,3\}) = \{3,2\} \notin \tau^\alpha = \tau^s$.

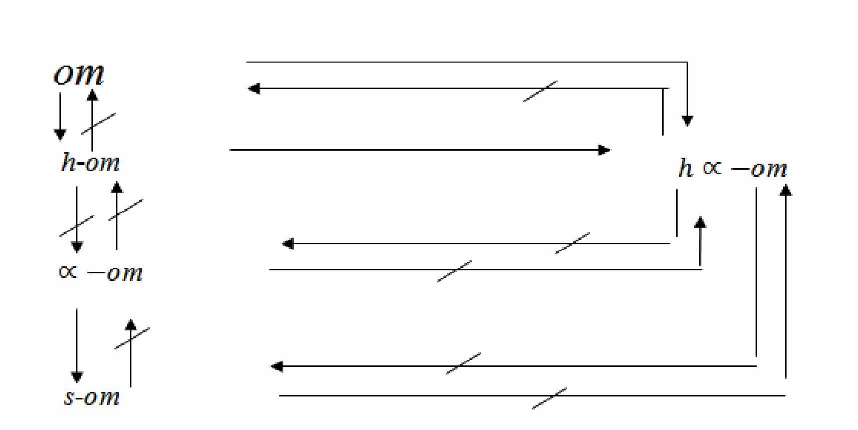

Fig3. The relationships of $h\alpha$ - open mappings with other open mappings

Definition3.13. A mapping $f: X \to Y$ is named "$h \propto$-irresolute" denoted by $(h\alpha - irem)$, if $f^{-1}(G)$ is ($h\propto$-os) in X, $\forall\, G \in$($h\propto$-os) in Y.

Example3.14. Let $X = Y = \{2,4,6\}$ and $\tau = \{\emptyset, X, \{4\}, \{4,6\}\}$,
$\tau^{h\propto} = \{\emptyset, X, \{4\}, \{6\}, \{2,6\}, \{4,6\}\}$, $\sigma = \{\emptyset, Y, \{4\}\}$,
$\tau^{h\propto} = \{\emptyset, Y, \{4\}, \{2,6\}\}$. Clearly, the identity mapping $f: X \to Y$ is $(h\alpha - irem)$.

Proposition3.15. Each $(h - irem)$ *is* $(h\alpha - irem)$.

Proof: Suppose $f: X \to Y$ be $(h - irem)$ and V be $(h - os)$ in Y. Because, f is $(h - irem)$, then, $f^{-1}(V)$ is$(h - os)$ in X. then, by lemma (2.3), $f^{-1}(V)$ is$(h\alpha - os)$in X. Henceforth, f is $(h\alpha - irem)$.

Remark3.16."The following example shows that $(h\alpha - irem)$ need not be$(s - irem)$ and$(\alpha - irem)$"

Example3.17. If $X = Y = \{1,3,5,7\}$ and $\tau = \{\emptyset, X, \{1\}, \{1,3\}, \{1,5\}, \{1,3,5\}\}$, then
$\tau^{h\propto} = \{\emptyset, X, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{1,3,5\}, \{3,5,7\}\}$
$\sigma = \{\emptyset, Y, \{1\}, \{3\}, \{1,3\}, \{1,3,5\}\}$, $\tau^{h\propto} = \{\emptyset, Y, \{1\}, \{3\}, \{1,3\}, \{1,3,5\}\}$.
The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha - irem)$ but it isn't $(s - irem)$ and *it* isn't $(\alpha - irem)$.

Proposition3.18. *Each (contm) is* $(h\alpha - irem)$.

Proof: Assume that" $f: X \to Y$ "be (*contm*) and V be any ($h\propto - os$) in Y. Since f is a continuous, then $f^{-1}(V)$ is (os) in X. Hence, $f^{-1}(V)$ is ($h\propto$-os) in X by "Lemma 2.4" . Henceforth, f is $(h\alpha - irem)$. ■

Proposition3.19. *Each* $(h\alpha - irem)$ *is* $(h\alpha - conm)$.

Proof: Assume that" $f: X \to Y$ "be $(h\alpha - irem)$ and V be any (os) in Y. Since each (os) is $(h\alpha - os)$ and since f is $h\propto$-irresolute, then $f^{-1}(V)$ is $(h\alpha - os)$ in X. Henceforth, f is $(h\alpha - conm)$. ■

The converse of 3.19 is not true. Indeed,
If $X = Y = \{2,4,6\}$ and $\tau = \{\emptyset, X, \{4\}, \{4,6\}\}$.
Then, $\tau^{h\propto} = \{\emptyset, X, \{6\}, \{4\}, \{2,6\}, \{4,6\}\}$, $\sigma = \{\emptyset, Y, \{6\}\}$, $\tau^{h\propto} = \{\emptyset, Y, \{6\}, \{2,4\}\}$.
The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha - conm)$ but f isn't $(h\alpha - irem)$ because for $(h\alpha - os)$ $\{2,4\}$, $f^{-1}(\{2,4\}) = \{2,4\}$ isn't $(h\alpha - os)$ in X.

Theorem3.20. *The composition of two($h\propto$-irem) is also($h\propto$-irem).*

Proof: Assume that" $f: X \to Y$ and $g: Y \to Z$" be any two ($h\propto - irem$). Let U be any ($h\propto$-os) in Z. Since g is ($h\propto - irem$) , then $g^{-1}(U)$ is ($h\propto$-os) in Y. Since f is ($h\propto - irem$) , then $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is ($h\propto$-os) in X. Henceforth, $gof: X \to Z$ is ($h\propto - irem$). ■

Theorem3.21. *If* " $f: X \to Y$ *is* ($h\propto$-*irem*) *and* $g: Y \to Z$" *is* $(h\alpha - conm)$, then $gof: X \to Z$ is ($h\propto - irem$).

Proof: Assume that " $f : X \to Y$ is ($h\propto$-*irem*) and $g{:}Y \to Z$" is ($h\alpha - conm$)". Let U be any (os) in Z. Then U is ($h\propto$-os) by Lemma 2.4. Since, g is ($h\alpha - conm$), then $g^{-1}(U)$ is ($h\propto$-os) in Y. Since f is ($h\propto - irem$), then $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ *is* ($h\propto$-*os*) in X. Henceforth, $gof{:}X \to Z$ is ($h\propto - irem$). ■

Definition3.22. A "bijective mapping" $f : X \to Y$ is named "$h\propto$-homeomorphism" denoted by ($h\alpha - homo$) if f is ($h\alpha - conm$) and ($h\alpha - om$).

Theorem3.23. *Each (homo) is* ($h\alpha - homo$)

Proof: Since each (*conm*) is ($h\alpha - conm$) by proposition (3.4). Also, since each (om) is ($h\alpha - om$) by proposition (3.10). Additionally, since *f* is bijective. Henceforth, *f* is ($h\alpha - homo$). ■

Theorem3.24. *Each* ($h - homo$) *is*($h\alpha - homo$)

Proof: Since each ($h - conm$) is ($h\alpha - conm$) by proposition (3.3). Also, since each ($h - om$) is ($h\alpha - om$) by proposition (3.9). Additionally, since *f* is bijective. Henceforth, *f* is ($h\alpha - homo$). ■

Remark3.25. "There is no relationship between ($h\alpha - homo$) with (s – *homo*) and (∝ - *homo*)"

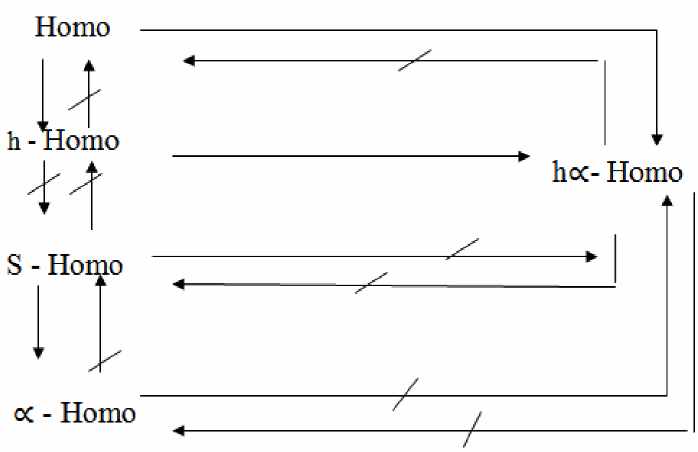

Fig4. The relationships of $h\alpha -$ homeomorphisms with other homeomorphisms mappings

4. *h* ∝-Totally Continuous Mapping

Definition4.1. A mapping $f : X \to Y$ is named "$h\propto$-totally continuous" denoted by ($h\alpha - tconm$), if the inverse image of each ($h\alpha - os$) of Y is ($cl - os$) in X.

Example4.2. Let $X = Y = \{1,3,5\}$ and $\tau = \{\emptyset, X, \{1\}, \{3,5\}\}$ $\sigma = \{\emptyset, Y, \{5\}\}$, $\sigma^{h\propto} = \{\emptyset, Y, \{5\}, \{1,3\}\}$.
$f : X \to Y$ defined by $f(5) = 1$, $f(3) = 3$, $f(1) = 5$. Obviously, f is ($h\alpha - tconm$).

Theorem4.3. Each ($h\alpha - tconm$) is ($tconm$).

Proof: Suppose that" $f : X \to Y$ " be ($h\alpha - tconm$) and *V* is (os) in *Y*, since each (os) is ($h\alpha - os$), then *V* is ($h\alpha - os$)in *Y*. Then, since *f* is ($h\alpha - tconm$), we get, $f^{-1}(V)$ is ($cl - os$) in *X*. Henceforth, *f* is ($tconm$). ■

Example4.4. If $X = Y = \{2,4,6\}$ and $\tau = \{\emptyset, X, \{2\}, \{4,6\}\}$ then $\sigma = \{\emptyset, Y, \{4,6\}\}$, $\tau^{h\propto} = \{\emptyset, Y, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}\}$.
The identity mapping $f{:}X \to Y$ is plainly apparent ($tconm$) but it isn't ($h\alpha - tconm$) because for ($h\alpha - os$) $\{2,4\}$, $f^{-1}(\{2,4\}) = \{2,4\}$ is not ($cl - os$).

Theorem4.5. *Each* ($h\alpha - tconm$) *is* ($h\alpha - irem$).

Proof: Assume that $f : X \to Y$ be ($h\alpha - tconm$) and *V* be ($h\alpha - os$) in *Y*. Then, since *f* is ($h\alpha - tconm$), we get $f^{-1}(V)$ is ($cl - os$) in *X*, it denotes, $f^{-1}(V)$ is (os), it follows $f^{-1}(V)$ is ($h\alpha - os$) in X. Henceforth, *f* is ($h\alpha - irem$). ■

Example4.6. If $X = Y = \{1,2,3\}$ and $\tau = \{\emptyset, X, \{1,2\}\}$ then $\tau^{h\propto} = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. $\sigma = \{\emptyset, Y, \{2\}\}$, $\sigma^{h\propto} = \{\emptyset, Y, \{2\}, \{1,3\}\}$. The identity mapping $f: X \to Y$ is plainly apparent $(h\alpha - irem)$ but f isn't $(h\alpha - tconm)$ because for $(h\alpha - os)$ $\{1,3\}, f^{-1}(\{1,3\}) = \{1,3\}$ is not $(cl - os)$ set in X.

Theorem4.7. The composition of two $(h\alpha - tconm)$ is also $(h\alpha - tconm)$.

Proof: Suppose that " $f : X \to Y$ and $g : Y \to Z$ " be $(h\alpha - tconm)$. Assume that V be$(h\alpha - os)$ in Z. Because, g is $(h\alpha - tconm)$, then $g^{-1}(V)$ is $(cl - os)$ in Y, it denotes $f^{-1}(V)$ (os), it follows $f^{-1}(V)$ is $(h\alpha - os)$. Then, since, f is $(h\alpha - tconm)$, we get, " $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ " is $(cl - os)$ in X. Henceforth, $gof: X \to Z$ is $(h\alpha - tconm)$. ■

Theorem4.8. *If " $f : X \to Y$ " be $(h\alpha - tconm)$ and " $g : Y \to Z$ " be $(h\alpha - irem)$, then " $g \circ f : X \to Z$ " is $(h\alpha - tconm)$.*

Proof: Assume that $f : X \to Y$ be $(h\alpha - tconm)$ and $g : Y \to Z$ is $(h\alpha - irem)$. Let W be $(h\alpha - os)$ in Z. Since g is $(h\alpha - irem)$ then $g^{-1}(W)$ is $(h\alpha - os)$ in Y. Since f is $(h\alpha - tconm)$, then $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is $(cl - os)$ in X. Henceforth, $g \circ f : X \to Z$ is $(h\alpha - tconm)$. ■

Theorem4.9. *If " $f : X \to Y$ "is $(h\alpha - tconm)$ and " $g : Y \to Z$ " is $(h\alpha - conm)$, then" $g \circ f : X \to Z$ is $(tconm)$".*

Proof: Assume that $f : X \to Y$ be $(h\alpha - tconm)$ and " $g : Y \to Z$ "is $(h\alpha - conm)$, let W be (os) in Z. Since, g is $(h\alpha - conm)$, then, $g^{-1}(W)$ is $(h\alpha - os)$ in Y. Since, f is $(h\alpha - tconm)$, then," $f^{-1}(g^{-1}(W)) = (gof)^{-1}(W)$ "is $(cl - os)$ in X. Henceforth, " $g \circ f : X \to Z$ "is $(tconm)$. ■

5. $h\propto$- Open Sets and Separating Axioms

Definition5.1. A TS (X, τ) is named

1. $T_{0h\alpha} - space$ if for any $m, n \in X$ with $m \neq n$ there exists $(h\alpha - os)$ U within either, $m \in U$ and $n \notin U$ or $n \in U$ and $m \notin U$.
2. $T_{1h\alpha}$-space if for any $m, n \in X$ with $m \neq n$ there exists $(h\alpha - os)$ U, V containing m, n respectively within either $n \notin U, m \notin V$.

Theorem.5.2. *Each T_0 - space is $T_{0h\alpha}$ space .*

Proof: Suppose that X is T_0 *space* , *m* and *n* are two distinct points in X. Since X is T_0 -space. Then there is (os) U in X within $m \in U$ and $n \notin U$ or $n \in U$ and $m \notin U$. Since each (os) is $(h\alpha - os)$. Then U is $(h\alpha - os)$ in X within $m \in U$ and $n \notin U$ or $n \in U$ and $m \notin U$. Henceforth X is $T_{0h\alpha}$- space. ■

Example.5.3. If $X = \{1,3,5\}$, $\tau = \{\emptyset, X, \{1,3\}\}$. Then (X, τ) isn't T_0- space, but $(X, \tau^{h\propto})$ is $T_{0h\alpha}$ - space.

Theorem.5.4. Each T_1-space is $T_{1h\alpha}$ -space.

Proof: Let X be a T_1- space and assume that two distinct points *m* and *n* in X. Since X is T_1- space. Then there exist two (os) U, V in X within $m \in U$, $n \notin U$ and $n \in V$ and $m \notin V$. Since each (os) is $(h\alpha - os)$. Then U, V are $(h\alpha - os)$ in X within $m \in U$ and $n \notin U$ and $n \in V$ and $m \notin V$. Henceforth X is $T_{1h\alpha}$ - space. ■

Example.5.5. Let $X = \{2,4,6\}$, $\tau = \{\emptyset, X, \{2\}, \{2,4\}, \{2,6\}\}$. Then (X, τ) isn't T_1- space, but $(X, \tau^{h\propto})$ is $T_{1h\alpha}$ - space.

Theorem5.6. *If " $f : X \to Y$ " be an injective, (h∝-irem) and Y is $T_{0\,h\propto} - space$, then X is also $T_{0\,h\propto} - space$.*

Proof: Assumes that $x, y \in X$ with $x \neq y$. Since f is injective and Y is $T_{0\,h\propto}$ − space there exists (h∝- *os*) U in Y s.t. $f(x) \in U$ and $f(y) \notin U$ or there exists (h∝- *os*) G in Y s.t. $f(y) \in G$ and $f(x) \notin G$ with *f(x)* $\neq$ *f(y)*. By (*h*∝-*irem*) of *f*, $f^{-1}(U)$ is (*h*∝-*os*) in X such that $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$ *or* $f^{-1}(G)$ is (*h*∝-*os*) in X such that $y \in f^{-1}(G)$ and $x \notin f^{-1}(G)$. This shows that X is $T_{0\ h\propto}-$ space. ■

Theorem5.7. *If " $f : X \to Y$ " be an injective, (h∝-irem) and Y is $T_{1\ h\propto}-space$, then X is also $T_{1\ h\propto}-space$.*

Proof: The argument exists in the similar way as mentioned in theorem 5.6 with suitable changes.

Theorem5.8. *If " $f : X \to Y$ " be bijection, (h∝-conm) and Y is $T_0-space$, then X is $T_{0h\propto}-space$.*

Proof: Let " $f : X \to Y$ " be bijection, (*h*∝-*conm*) and Y is T_0- space. to proof that X is a $T_{0h\propto}-$ space. Let $x_1,\ x_2 \in X$with $x_1 \neq x_2$. Since f is a bijection, there exists $y_1, y_2\ \in Y$with $y_1 \neq\ y_2$ such that $f(x_1)=\ y_1$ and $f\,(x_2)\ =\ y_2$, then $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since Y is T_0- space**,** there exists an (*os*) U in X such that $y_1\ \in U$ and $y_2 \notin U$. Since f is (*h*∝- *conm*), $f^{-1}(U)$ is a (*h*∝-*os*) in Y. Now we have $y_1\ \in U$ then $f^{-1}(y_1) \subset f^{-1}(U)$ then $x_1 \in f^{-1}(U)$ and $x_2 \notin f^{-1}$(U). Hence for any two distinct points $y_1,\ y_2$ in Y, there exists (*h*∝-*os*) $f^{-1}(U)$ in Y such that $x_1 \in f^{-1}(U)$ and $x_2 \notin f^{-1}(U)$. Hence (X,τ) is a $T_{0h\propto}-$ space. ■

Theorem5.9. *If " $f : X \to Y$ "be injective,(h∝-conm) and Y is $T_1-space$, then X is $T_{1h\propto}-space$.*

Proof: Suppose *x*, *y*$\in X$. Such that *x*$\neq$ *y*. Since f is injective then $f\ (x) \neq f\ (y)$. Since Y *is* $T_{1h\propto}-$ space then there are two (*h* ∝- *os*) U and V in Y s.t. $f\ (x)\ \in U$, $f\ (y) \notin U$ and $f\ (y)\ \in V$, $f\ (x) \notin V$. Since f is (*h* ∝- *conm*) then $f^{-1}(U)$, $f^{-1}(V)$ are two (*h* ∝-*os*) in X , $x\in f^{-1}(U)$, $y\notin f^{-1}(U)$ and $y\in f^{-1}(V)$, $x\notin f^{-1}(V)$. Hence (X,τ) is $T_{1h\propto}-$ space. ■

Conclusions: It is concluded in this work that each h-open set is *h* ∝-open, each open set is *h* ∝-open set and there is no relationship between∝-open sets and semi-open sets with *h* ∝-open sets. Furthermore, each h-continuous mapping is *h* ∝-continuous mapping, each continuous mapping is *h* ∝-continuous mapping and there is no relationship between∝-continuous mappings and semi-continuous mappings with *h* ∝-continuous mappings**.**

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