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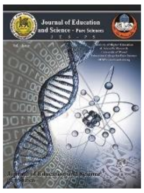

Haar Wavelet Method For the Numerical Solution of Nonlinear Fredholm Integro-Differential Equations

Najem A. Mohammad[1] Younis A. Sabawi[1,2] * Mohammad Sh. Hasso[1]

[1] Department of Mathematics, Faculty of Science and Health, Koya University, Koya, Kurdistan Region - F.R. Iraq
[2] Department of mathematics Education, Faculty of Education, Tishk International University, Erbil, Iraq

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Correspondence:
Najem A. Mohammad1
najem.mohammad@koyauniversity.org

Abstract

The solution of nonlinear Fredholm integro-differential equations plays a significant role in analyzing many nonlinear events that occur in chemistry, physics, mathematical biology, and a variety of other fields of science and engineering. A physical event can be represented by a differential equation, an integro-differential equation since many of these equations cannot be solved directly or it is difficult to solve. Numerical approaches that are useful combinations of numerical integration must frequently be used. This work presents a method for solving the type of nonlinear Fredholm integro-differential equation (NFIDE) of the second kind. The Leibnitz rule is used with the Haar wavelet collocation method in this paper to solve NFIDE numerically. Some techniques are used to transfer the equation into an algebraic system through an operational matrix. The convergence analysis had been proved through this work and the numerical experiments had been given to illustrate the effectiveness of the proposed method based on MATLAB programming.

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1. Introduction

The main text formatIntegro-differential equations have several applications in mathematics, science, and technology. Integro-differential equations appear often in fluid mechanics, biological models, solid-state physics, kinetics in chemistry, and so on. Most problems are difficult to solve, especially analytically[1-5]. However, analytical solutions to Integro-differential equations do not exist or are difficult to find. Therefore, various numerical approaches for solving Integro-differential equations have been developed.

Consider the Fredholm integro-differential equation as the form

$$y''(x) = f(x) + \lambda \int_a^b k(x,t)G\big(y(t)\big)dt, \qquad (1)$$

with initial conditions
$y(a) = \alpha, y'(a) = \beta,$

where the function $f \in L^2[a,b)$ and the kernel $k \in L^2[a,b)$, and $G\big(y(t)\big)$ is a nonlinear function defined on the interval $[a,b]$, the solution of the integro-differential equation is denoted by the unknown function $y(x)$.

Wavelet theory is a relatively new and still developing method in applied mathematics. Since 1991, many kinds of wavelet methods have been used to solve different kinds of integro-differential equations numerically, and the possibility is to apply the method of Haar wavelets, which are the mathematically easiest wavelets. Haar wavelets have also been utilized for solving the two-dimensional nonlinear mixed Volterra-Fredholm-Hammerstein integral equation as well as the delay Volterra-Fredholm integral equation[6-7].

The Haar wavelet technique was employed by Lepik. [8], Babolian. [9], and Aziz et al. [10] to solve nonlinear Fredholm integral equations. In fact, the application of the Leibnitz rule based on the Haar wavelet collocation technique in numerical analysis is not new, and the Haar wavelets provide a number of benefits, such as simplicity, orthogonality, and extremely compact support. Sparse representation, fast transformation, and the ability to generate a quick algorithm for matrix representation are the major benefits of the Haar wavelets approach. Because the Haar basis is the simplest instance of a spline wavelet, resulting in the polynomial degree being set to zero, the computing costs of Haar wavelets are extremely minimal. So, we utilize them to solve a number of problems by converting them into a system of non-linear equations at collocation points and solving them with Matlab. In [11-13], Haar wavelet approaches are applied to different problems and fields.

The present study is prepared as follows: The Haar wavelet properties and definitions are covered in section 2. In sections 3 and 4, the convergence analysis of the Haar wavelets and the Haar wavelet operational matrix and its integrals are given respectively. In section 5, the solution procedure is presented. Finally, some numerical experiments with conclusions are shown in sections 6 and 7, respectively.

2. Haar Wavelet Properties

Among the different wavelet kinds, Haar wavelets are the most basic. They are real-line step functions that can only accept the numbers 0, 1, and (-1). We used the Haar wavelet method because it is fast, easy, flexible, and simple to compute. A family of switching rectangular wave forms with variable amplitudes is known as the Haar functions. Haar wavelets are generally defined on the interval $[0,1]$, but in many applications, they are formed on the interval $[a,b]$. The interval $[a,b]$ is subdivided into m equal parts. In this case, the orthogonal set of Haar functions is defined as [14], [15] on the interval $[a,b]$.

2.1. Haar wavelets and their definitions

Haar wavelets are created by integrating pairs of piecewise constant functions. Furthermore, the Haar functions are orthogonal, which makes them an excellent transform basis. Because of discontinuities at breaking points, the Haar functions are not differentiable.

Suppose that the integration interval $[a,b]$ is partitioned into 2^{J+1} subintervals of equal length, where $\Delta x = \frac{b-a}{2^{J+1}}. J \in N$ represents the highest degree of resolution. The translation and dilation parameters are indicated by $j = 0,1,2,\dots,J$ and $k = 0,1,2,\dots 2^j - 1$ correspondingly. The Haar family is defined a

$$h_i(x) = \begin{cases} 1 & \text{if } x \in [\xi_1, \xi_2) \\ -1 & \text{if } x \in \{\xi_2, \xi_3) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $\xi_1 = \frac{k}{m}, \xi_2 = \frac{k+0.5}{m}$, and $\xi_3 = \frac{k+1}{m}$, while $m = 2^j$.

The index i in Eq.(2) is evaluated by using $i = m + k + 1$, and in the case of minimal values $m = 1, k = 0$, then $i = 2$. The maximal value of i is $N = 2^{J+1}$. For $i \geq 2$, Eq.(2) is accurate.
The Haar scaling function at the value of $i = 1$ is defined as follows:

$$h_1(x) = \begin{cases} 1 & \text{if } x \in [a,b) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The collocation Points is given by $x_l = \frac{l-0.5}{N}$, where $l = 1,2,\dots,N$, and $H(i,l) = h_i(x_l)$ is the Haar coefficient matrix which is a square matrix of the dimension $N \times N$. If we take $J = 3$ we obtain the Harr coefficient matrix of order 16 which is defined as follows:

$$H(16,16)=\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&-1&-1\\
1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1
\end{bmatrix}.$$

2.2 Approximation of function by Haar wavelets

If $f(x)$ is any function on $[a,b]$ with finite energy and square intelligibility, then $f\in L^2[a,b)$ can be represented as an infinite sum of Haar wavelets

$$f(x)=\sum_{i=1}^{\infty}a_ih_i(x),$$

The Haar coefficients are represented as a_i. The series can be truncated to finite terms if f is a piecewise constant or must be approximated by a piecewise constant at each subinterval, as shown below.

$$f(x)=\sum_{i=1}^{m}a_ih_i(x).$$

2.3 Convergence analysis of the Haar wavelets

In this section, we will discuss the convergence analysis of the Haar wavelets by integration Eq.(2), from 0 to x, and using the initial conditions with properties of approximation of the functions of Haar wavelets, then we have

$$y(x)=\alpha+\beta x+\int_0^x(x-t)\left(\sum_{n=0}^{\infty}a_nh_n(t)\right)dt+\int_0^x(x-t)\left(\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}a_ib_jh_i(x)h_j(t)\right)\left(\sum_{n=0}^{k-1}\widetilde{a_n}\,h_n(t)\right).$$

Now we suppose that $y(x)\in C'[0,1]$ with $|y'(x)|\le L$, for each $x\in(0,1)$. To approximate the function $y(x)$ we suppose the following relation

$$y_k(x)=\sum_{n=0}^{k-1}a_n\,h_n(t),$$

where $k=2^{(r+1)}, r=0,1,2,\ldots$, then

$$y(x)-y_k(x)=\sum_{n=2^{r+1}}^{\infty}a_n\,h_n(t).$$

So, we construct the norm of the given equation as follows:

$$\|y(x)-y_k(x)\|_{L^2[0,1)}=\left(\int_0^1(|a_n(x)-u_k(x)|^2)dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^1 \sum_{n=2^{r+1}}^{\infty} a_n h_n(x) \sum_{\hat{a}=2^{r+1}}^{\infty} \widehat{a_n} h_n(x) dx \right)^{\frac{1}{2}}$$

$$= \left(\sum_{n=2^{r+1}}^{\infty} \sum_{\hat{n}=2^{r+1}}^{\infty} a_n \int_0^1 h_n(x) h_{\hat{n}}(x) dx \right)^{\frac{1}{2}}$$

$$= \sum_{n=2^{r+1}}^{\infty} (a_n)^{\frac{1}{2}}.$$

Therefore

$$\|y(x) - y_k(x)\|_{L^2[0,1)}^2 = \sum_{n=2^{r+1}}^{\infty} a_n^2, \tag{4}$$

where a_n is the inner product between $y(x)$ and $h_n(x)$.
By substituting $\psi_n(x) = h_n(x) = m^{\frac{1}{2}} H(mx - k), k = 0,1, \dots, m - 1$, in inner product of a_n, we get

$$a_n = \int_0^1 m^{\frac{1}{2}} H(mx - k) y(x) dx \tag{5}$$

Substituting Eq.(2) in Eq.(5) we obtain

$$a_n = m^{\frac{1}{2}} \int_{\frac{k}{m}}^{\frac{k+0.5}{m}} y(x) dx - \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} y(x) dx.$$

There exist x_1 and x_2 according to the mean value theorem

$$\frac{k}{m} \le x_1 < \frac{k + 0.5}{m} \; and \; \frac{k + 0.5}{m} \le x_2 < \frac{k + 1}{m},$$

then

$$a_n = m^{\frac{1}{2}} \left\{ \left(\frac{k + 0.5}{m} \right) y(x_1) - \left(\frac{k + 1}{m} - \frac{k + 0.5}{m} \right) y(x_2) \right\}.$$

By simplification the above equation, we obtain

$$a_n = \frac{1}{2\sqrt{m}} \big(y(x_1) - y(x_2) \big),$$

therefore

$$a_n^2 = \frac{1}{2m} \big(y(x_1) - y(x_2) \big)^2$$

and

$$a_n^2 = \frac{1}{2m} (x_2 - x_1)^2 y'^2(x_0) \le \frac{1}{4m^3} B^2.$$

From Eq.(4), we have

$$\|y(x) - y_k(x)\|_{L^2[0,1)}^2 = \sum_{n=k}^{\infty} a_n^2$$

$$= \sum_{j=r+1}^{\infty} \left(\sum_{n=m}^{2m-1} a_n^2 \right),$$

$$\le \sum_{j=r+1}^{\infty} \left(\sum_{n=m}^{2m-1} \frac{1}{4m^3} B^2 \right),$$

$= B^2 \sum_{j=r+1}^{\infty} \frac{1}{4m^2},$

$= \frac{B^2}{3} \frac{1}{k^2}.$

Therefore

$\|y(x) - y_k(x)\|_{L^2[0,1)}^2 = O\left(\frac{1}{k}\right).$

2.4 The Haar wavelet operational matrix and its integrals

The following equation

$$P_{1,i}(x) = \int_0^x h_i(t)dt$$

defines the operational matrix of integration, which is $N \times N$ square matrix. The form

$$P_{s+1,i}(x) = \int_0^x P_{s,i}(t)dt\,, where\;\; i = 1,2, \dots N$$

and $s = 1,2, \dots, n$ is used to develop a general operational matrix. According to the function $P_{1,i}(x)$ and using Eq.(2) contracted the following integrals

$$P_{1,i}(x) = \begin{cases} x - \xi_1 & \text{if } x \in [\xi_1, \xi_2) \\ \xi_3 - x & \text{if } x \in \{\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

$$P_{2,i}(x) = \begin{cases} \frac{1}{2!}(x - \xi_1)^2 & \text{if } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2!}(\xi_3 - x)^2 & \text{if } x \in \{\xi_2, \xi_3) \\ \frac{1}{4m^2} & \text{if } x \in [\xi_3, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

$$P_{3,i}(x) = \begin{cases} \frac{1}{6}(x - \xi_1)^2 & \text{if } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2}(x - \xi_2) - \frac{1}{6}(\xi_3 - x)^3 & \text{if } x \in \{\xi_2, \xi_3) \\ \frac{1}{4m^2}\;(x - \xi_2) & \text{if } x \in [\xi_3, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

and the fourth integrals can be formed as the followin

$$P_{4,i}(x) = \begin{cases} \frac{1}{24}(x - \xi_1)^4 & \text{if } x \in [\xi_1, \xi_2) \\ \frac{1}{8m^2}(x - \xi_2)^2 - \frac{1}{24}(\xi_3 - x)^4 + \frac{1}{192m^4} & \text{if } x \in \{\xi_2, \xi_3) \\ \frac{1}{8m^2}\;(x - \xi_2)^2 + \frac{1}{192m^4} & \text{if } x \in [\xi_3, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

The generalized integrals of Haar functions of the order n are considered as below:

$$P_{s,i}(x) = \begin{cases} \frac{1}{s!}(x - \xi_1)^s & \text{if } x \in [\xi_1, \xi_2) \\ \frac{1}{s!}\{(x - \xi_1)^s - 2(x - \xi_2)^s\} & \text{if } x \in \{\xi_2, \xi_3) \\ \frac{1}{s!}\{(x - \xi_1)^s - 2(x - \xi_2)^s + (x - \xi_3)^s\} & \text{if } x \in [\xi_3, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

If we choose different values for J, for example, if we take $J = 2$, then $N = 8$, so we get the following matrices

$$P_{1,i}(8,8)\frac{1}{16}=\begin{bmatrix}1&3&5&7&9&11&13&15\\1&3&5&7&7&5&3&1\\1&3&3&1&0&0&0&0\\0&0&0&0&1&3&3&1\\1&1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&1\end{bmatrix}$$

and

$$P_{2,i}(8,8)\frac{1}{16}=\begin{bmatrix}1/32&9/32&25/32&49/32&81/32&121/32&169/32&225/32\\1/32&9/32&25/32&49/32&79/32&103/32&119/32&127/32\\1/32&9/32&23/32&31/32&1&1&1&1\\0&0&0&0&1/32&9/32&23/32&31/32\\1/32&7/32&1/4&1/4&1/4&1/32&1/4&1/4\\0&0&1/32&7/32&1/4&1/32&1/4&1/4\\0&0&0&0&1/32&7/32&1/4&1/4\\0&0&0&0&0&0&1/32&7/32\end{bmatrix}$$

But if we select $J=3$, then $N=16$, we obtain the following matrices

$$P_{1,i}(16,16)\frac{1}{32}=\begin{bmatrix}1&3&5&7&9&11&13&15&17&19&21&23&25&27&29&31\\1&3&5&7&9&11&13&15&15&13&11&9&7&5&3&1\\1&3&5&7&7&5&3&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&3&5&7&7&5&3&1\\1&3&3&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&3&3&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&3&3&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&3&3&1\\1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\end{bmatrix}$$

and $P_{2,i}(16,16)$ is given as

$$=\frac{1}{2048}\begin{bmatrix}1&9&25&49&81&121&169&225&289&361&441&529&625&729&841&961\\1&9&25&49&81&121&169&225&287&343&391&431&463&487&503&511\\1&9&25&49&79&103&119&127&128&128&128&128&128&128&128&128\\0&0&0&0&0&0&0&0&1&9&25&49&79&103&119&127\\1&9&23&31&32&32&32&32&32&32&32&32&32&32&32&32\\0&9&0&0&1&9&23&31&32&32&32&32&32&32&32&32\\0&9&0&0&0&0&0&0&1&9&23&31&32&32&32&32\\0&9&0&0&0&0&0&0&0&0&0&0&1&9&23&31\\1&7&8&8&8&8&8&8&8&8&8&8&8&8&8&8\\0&0&1&7&8&8&8&8&8&8&8&8&8&8&8&8\\0&0&0&0&1&7&8&8&8&8&8&8&8&8&8&8\\0&0&0&0&0&0&1&7&8&8&8&8&8&8&8&8\\0&0&0&0&0&0&0&0&1&7&8&8&8&8&8&8\\0&0&0&0&0&0&0&0&0&0&1&7&8&8&8&8\\0&0&0&0&0&0&0&0&0&0&0&0&1&7&8&8\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&7\end{bmatrix}.$$

2.5 The solution procedure

This section introduced a Leibnitz-Haar wavelet collocation technique for solving a second-order nonlinear Fredholm integro-differential equation by reducing it to an equivalent differential equation with initial conditions. The Leibnitz rule for integral differentiation is used in the conversion [16].

Consider the following integral equation:

$$F(x) = \lambda \int_{g_1(x)}^{g_2(x)} k(x,t)y(t)dt \qquad (11)$$

therefore, the differentiation of integrals exists in Eq.(1) and is derived by

$$F'(x) = \frac{dF}{dx} k(x, g_2(x))(y(g_2(x))\frac{dg_2(x)}{dx} - k(x, g_1(x))(y(g_1(x)) + \lambda \int_{g_1(x)}^{g_2(x)} \frac{\partial k(x,t)}{\partial x} y(t)dt, \qquad (12)$$

if $g_1(x) = a$ and $g_2(x) = b$, where a and b are constants, then the Leibnitz rule for Eq.(11) reduces to

$$F'(x) = \frac{dF}{dx} = \lambda \int_{g_1(x)}^{g_2(x)} \frac{\partial k(x,t)}{\partial x} y(t)dt \qquad (13)$$

The numerical computing technique has been as described in the following: The first step, we differentiate Eq.(1) with regard to x by using the Leibnitz rule, then we obtain differential equations as shown below:

$$y'''(x) = f'(x) + \lambda F'(x)$$
$$y^{(iv)}(x) = f''(x) + \lambda F''(x)$$
$$\vdots \qquad (14)$$
$$y^{(n)}(x) = f^{(n-2)}(x) + \lambda F^{(n-2)}(x)$$

and so on until $F^{(n)} = 0, n = 3,4, \dots$ with initial conditions $y(0) = \alpha, y'(0) = \beta, y''(0) = \gamma, y'''(0) = \delta, \dots y^{(n-1)}(0) = \eta$.

In the second step, we suppose that

$$y^{(n)}(x) = \sum_{i=1}^{N} a_i h_i(x), \qquad (15)$$

and using the Haar wavelets collocation approach.

The third step is to integrate Eq.(15) until we achieve the approximate solution, as shown below

$$y^{(n-1)}(x) = \delta + \sum_{i=1}^{N} a_i P_{1,i}(x) \qquad (16)$$

$$y^{(n-2)}(x) = \gamma + \delta x + \sum_{i=1}^{N} a_i P_{2,i}(x) \qquad (17)$$

$$y^{(n-3)}(x) = \beta + \gamma x + \delta \frac{x^2}{2} + \sum_{i=1}^{N} a_i P_{3,i}(x) \qquad (18)$$

$$\vdots$$

$$y(x) = \alpha + \beta x + \gamma \frac{x^2}{2!} + \delta \frac{x^3}{3!} + \cdots + \eta \frac{x^n}{n!} + \sum_{i=1}^{N} a_i P_{n,i}(x) \qquad (19)$$

The final step is to replace Eq.(15)-Eq.(19) in Eq.(14), which reduces the nonlinear system of N equations with N unknowns, and then employ Newton's technique to obtain the Haar coefficients a_i, Where $i = 1, 2, \dots, N$. Eventually replacing Haar coefficients in Eq.(19) to determine the suitable approximation solutions of Eq.(1).

3. Numerical Experiments

This section illustrates the performance of a proposed method, through an implementation based on Matlab programming. The pointwise error is used to measure the error between the numerical and analytical solutions. We denote by E errors term described by

$E_{HWM} = y(x) - Y_{HWM}.(x).$

Let us introduce the three accuracy indicators when using space step size h.

The pointwise error

$\varepsilon_{HWM} = |E(x_i)|$

The l^∞ norm of the error

$l^\infty(E_{HWM}, h) = \max_{0\le i\le N} |E_{HWM}(x_i)|$

The l^2 norm of the error

$$l^2(E_{HWM}, h) = \sqrt{h\sum_{i=0}^{N} |E_{HWM}(x_i)|^2}.$$

Example 3.1. Consider second order nonlinear Fredholm integro-differential equation

$$y''(x) = -sinx - \frac{\pi x}{16} + \frac{\pi^2}{32} + \frac{1}{8}\int_0^\pi (x-t)y^2(t)dt \qquad (20)$$

with initial conditions $y(0) = 1, y'(0) = 0$, and exact solution $y(x) = sinx$.

Differentiating the Eq.(20) with respect to x, gives

$$y'''(x) = -cosx - \frac{\pi}{16} + \frac{1}{8}\int_0^\pi y^2(t)dt,$$

we have the following differential equation by differentiating the above equation again according to x,

$$y^{(iv)}(x) = sinx, \qquad (21)$$

with initial conditions $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1$.

Using the Haar wavelet approach, we suppose that

$$y^{(iv)}(x) = \sum_{i=1}^{N} a_i h_i(x). \qquad (22)$$

The approximate solution can be illustrated below after integrating Eq.(22), and applying the initial conditions.

$$y'''(x) = -1 + \sum_{i=1}^{N} a_i P_{1,i}(x) \qquad (23)$$

$$y''(x) = -x + \sum_{i=1}^{N} a_i P_{2,i}(x) \qquad (24)$$

$$y'(x) = 1 - \frac{x^2}{2} + \sum_{i=1}^{N} a_i P_{3,i}(x) \qquad (25)$$

$$y(x) = x - \frac{x^3}{3!} + \sum_{i=1}^{N} a_i P_{4,i}(x). \qquad (26)$$

The approximate solution of Eq.(20) is given by Eq.(26). Now by replacing Eq.(22) in the differential Eq.(21), then reduce the nonlinear system as follows

$$\sum_{i=1}^{N} a_i h_i(x) = sinx, \qquad (27)$$

Newton's technique was employed to get the Haar coefficients a_i, in Eq.(27). Finally, we substitute the values of Haar coefficients a_i, in Eq.(26), we get the approximate solution.

Table 1. Comparison between the exact solution and approximate solution of Example 3.1

$x_i(/32)$	$y(x)$	$Y_{HWM}(x)$	ε_{HWM}	l_{HWM}^{∞}	l_{HWM}^{2}
1	0.031245	0.031245	9.9321e-10		
3	0.093613	0.093613	4.2708e-08		
5	0.15561	0.15562	2.0353e-07		
7	0.21701	0.21701	5.626e-07		
9	0.27756	0.27756	1.1987e-06		
11	0.33702	0.33702	2.19e-06		
13	0.39517	0.39517	3.6141e-06		
15	0.45177	0.45178	5.5476e-06	4.8174e-05	2.1289e-04
17	0.50661	0.50662	8.0664e-06		
19	0.55947	0.55948	1.1245e-05		
21	0.61015	0.61017	1.5156e-05		
23	0.65844	0.65846	1.9873e-05		
25	0.70417	0.70419	2.5465e-05		
27	0.74714	0.74717	3.2002e-05		
29	0.7872	0.78724	3.955e-05		
31	0.82418	0.82423	4.8174e-05		

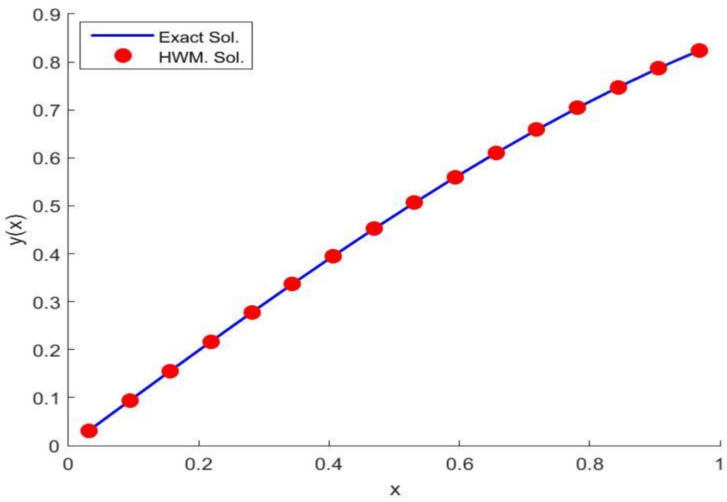

Figure 1. Comparison between the exact solution and approximate solution of Example 3.1

Example 3.2. Consider second-order nonlinear Fredholm integro-differential equation

$$y''(x) + xy'(x) - xy(x) = e^x - sinx + \int_0^1 sinx\, e^{-2t} y^2(t)dt \tag{28}$$

with initial conditions $y(0) = 1, y'(0) = 1$, an analytical solution is $y(x) = e^x$.

Differentiating the Eq.(28) twice with respect to x, gives

$$y'''(x) + xy''(x) + y'(x) - xy'(x) - y(x) = e^x - cosx + \int_0^1 cosx\, e^{-2t} y^2(t)dt \tag{29}$$

and

$$y^{iv}(x) + xy'''(x) + 2y''(x) - xy''(x) - 2y'(x) = e^x + sinx - \int_0^1 sinx\, e^{-2t} y^2(t)dt \tag{30}$$

from Eq.(28), we have

$$-\int_0^1 sinx\, e^{-2t} y^2(t)dt = e^x - sinx - y''(x) - xy'(x) + xy(x) \tag{31}$$

Replacing Eq.(30) into Eq.(31), we obtain

$$y^{iv}(x) + xy'''(x) + 2y''(x) - xy''(x) - 2y'(x) = e^x + sinx + e^x - sinx - y''(x) - xy'(x) + xy(x)$$

$$y^{iv}(x) + xy'''(x) + 3y''(x) - xy''(x) - 2y'(x) + xy'(x) - xy(x) = 2e^x \qquad (32).$$

By simplification the Eq.(32), we get the differential equation as the following form

$$y^{iv}(x) + xy'''(x) + (3 - x)y''(x) - (2 - x)y'(x) - xy(x) = 2e^x \tag{33}$$

with initial conditions

$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1.$

Using the Haar wavelet approach, we suppose that

$$y^{(iv)}(x) = \sum_{i=1}^{N} a_i h_i(x). \qquad (34)$$

The approximate solution can be found below after integrating Eq.(34), and applying the initial conditions.

$$y'''(x) = 1 + \sum_{i=1}^{N} a_i P_{1,i}(x) \qquad (35)$$

$$y''(x) = 1 + x + \sum_{i=1}^{N} a_i P_{2,i}(x) \qquad (36)$$

$$y'(x) = 1 + x + \frac{x^2}{2} + \sum_{i=1}^{N} a_i P_{3,i}(x) \qquad (37)$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \sum_{i=1}^{N} a_i P_{4,i}(x). \qquad (38)$$

The approximate solution of Eq.(28) is given by Eq.(38). Now by replacing Eq.(34) in the differential Eq.(31), then reduce the nonlinear system as the following form:

$$\sum_{i=1}^{N} a_i h_i(x) + x\left(\sum_{i=1}^{N} a_i P_{2,i}(x)\right) + (3-x)\left(1 + x + \sum_{i=1}^{N} a_i P_{2,i}(x)\right) - (2-x)\left(1 + x + \frac{x^2}{2} + \sum_{i=1}^{N} a_i P_{3,i}(x)\right) - x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \sum_{i=1}^{N} a_i P_{4,i}(x)\right) = 2e^x. \qquad (39)$$

Newton's technique was employed to get the Haar coefficients a_i in Eq.(39). Finally, we substitute the values of Haar coefficients a_i in Eq.(38), we get the approximate solution.

Table 2. Comparison between the exact solution and approximate solution of Example 3.2.

$x_i(/32)$	$y(x)$	$Y_{HWM}(x)$	ε_{HWM}	l_{HWM}^{∞}	l_{HWM}^{2}
1	1.0317	1.0317	2.2892e-09	0.0090	0.0275
3	1.0983	1.0983	1.4976e-07		
5	1.1691	1.1691	1.2346e-06		
7	1.2445	1.2445	5.6305e-06		
9	1.3248	1.3248	1.8454e-05		
11	1.4102	1.4103	4.8817e-05		
13	1.5012	1.5013	0.00011115		
15	1.598	1.5982	0.00022661		
17	1.7011	1.7015	0.00042447		
19	1.8108	1.8115	0.00074366		
21	1.9276	1.9288	0.0012342		
23	2.0519	2.0538	0.0019586		
25	2.1842	2.1872	0.0029936		
27	2.3251	2.3295	0.0044314		
29	2.475	2.4814	0.006381		
31	2.6346	2.6436	0.0089698		

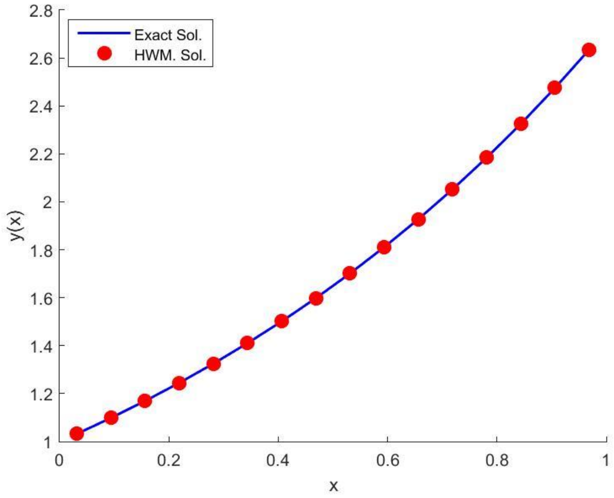

Figure 2. Comparison between exact solution and approximate solution of Example 3. 2.

Example 3.3. Consider second-order nonlinear Fredholm integro-differential equation

$$y''(x) = e^x + \frac{1}{4}(e^2 - 2)x + \frac{1}{2}\int_0^1 x(t - y^2)(t)dt, \qquad (40)$$

with initial conditions $y(0) = 1, y'(0) = 1$, and exact solution $y(x) = e^x$.

Differentiating the Eq.(40) with respect to x, gives

$$y'''(x) = e^x + \frac{1}{4}(e^2 - 2) + \frac{1}{2}\int_0^1 (t - y^2)(t)dt.$$

We have the following differential equation by differentiating the above equation again according to x,

$$y^{(iv)}(x) = e^x, \qquad (41)$$

with initial conditions $y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1$.

Using the Haar wavelets approach, we suppose that

$$y^{(iv)}(x) = \sum_{i=1}^{N} a_i h_i(x) \qquad (42)$$

The approximate solution can be found below after integrating Eq.(40), and applying the initial conditions.

$$y'''(x) = 1 + \sum_{i=1}^{N} a_i P_{1,i}(x) \qquad (43)$$

$$y''(x) = 1 + x + \sum_{i=1}^{N} a_i P_{2,i}(x) \qquad (44)$$

$$y'(x) = 1 + x + \frac{x^2}{2} + \sum_{i=1}^{N} a_i P_{3,i}(x) \qquad (45)$$

$$\vdots$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \sum_{i=1}^{N} a_i P_{4,i}(x). \qquad (46)$$

The approximate solution of Eq.(40) is given by Eq.(46). Applying Eq.(42) to the differential Eq.(41) reduces the nonlinear system as follows:

$$\sum_{i=1}^{N} a_i h_i(x) - e^x = 0, \qquad (47)$$

then Newton's technique employed to get the Haar coefficients a_i in Eq.(47). Finally, we substitute the values of Haar coefficients a_i in Eq.(46), we obtain the approximate solution.

Table 3. Comparison between the exact solution and approximate solution of Example 3.3.

$x_i(/32)$	$y(x)$	$Y_{HWM}(x)$	ε_{HWM}	l_{HWM}^{∞}	l_{HWM}^{2}
1	1.0317	1.0317	1.0117e-09	5.6679e-05	2.4240e-04
3	1.0983	1.0983	4.351e-08		
5	1.1691	1.1691	2.0857e-07		
7	1.2445	1.2445	5.8098e-07		
9	1.3248	1.3248	1.2484e-06		
11	1.4102	1.4102	2.3017e-06		
13	1.5012	1.5012	3.8349e-06		
15	1.598	1.598	5.9455e-06		
17	1.7011	1.7011	8.7349e-06		
19	1.8108	1.8108	1.2308e-05		
21	1.927	1.9276	1.6775e-05		
23	2.0519	2.0519	2.225e-05		
25	2.1842	2.1842	2.8851e-05		
27	2.3251	2.3251	3.6702e-05		
29	2.475	2.4751	4.5933e-05		
31	2.6346	2.6347	5.6679e-05		

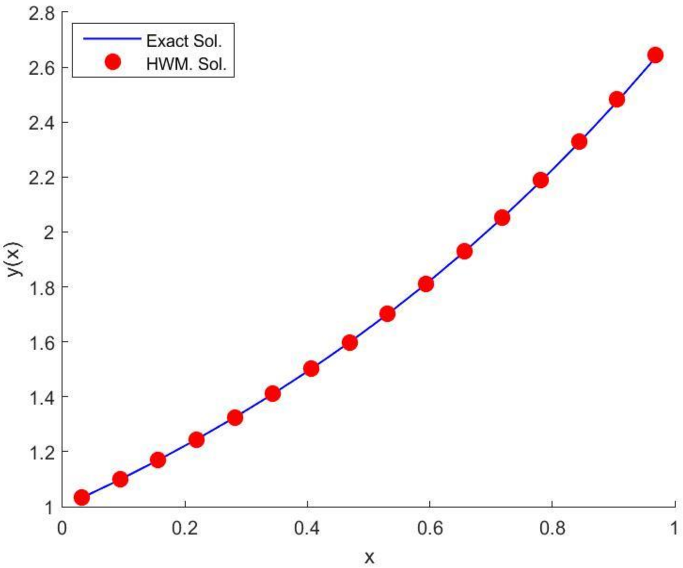

Figure 3. Comparison between the exact solution and approximate solution of Example 3.3.

4. Conclusion

The numerical solution of nonlinear Fredholm integro-differential equations of the second order was obtained in this work using the Haar wavelet collocation method based on the Leibnitz rule. To solve the resulting integral equations, the Haar wavelet function and its operational matrix were also applied. The integro-differential equations are reduced to a set of algebraic equations after being reduced to differential equations with initial conditions. The error analysis reveals that the approximation becomes more accurate as the level of resolution N is raised. As a result, a bigger N is advised for better results. In the future, this work can be solved by using finite element methods. For more details, see [17-24]. Another interesting direction would be the using finite difference and compact finite difference methods, see [25-28].

5. Acknowledgements

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طريقة مويجات هار للحل العددي لمعادلات فريدهولم التفاضلية التكاملية غير الخطية

نجم عبدالله محم[11] يونس عبد سبعاوي [2,1]* محمد شامي حسو [1]

[1] قسم علوم الرياضيات، كلية العلوم والصحة، جامعة كويه>،كورد ستان العراق , العراق.
[2] قسم تربية الرياضيات، كلية التربية ، جامعة تشك الدوليه، كورد ستان العراق , العراق

الخلاصة:
ان حل معادلات فريدهولم التفاضلية التكاملية غير الخطية تلعب دورًا مهمًا في تحليل العديد من الأحداث غير الخطية التي تظهر في الكيمياء والفيزياء والبيولوجيا الرياضية ومجموعة متنوعة من مجالات العلوم والهندسة الأخرى. ان المسائل الفيزيائية يمكن وصفها كمعادلة تفاضلية ، معادلة تكاملية ومعادلة تفاضلية تكاملية. يجب إستخدام الأساليب العددية التي تعتبر مجموعات مفيدة للتكامل العددي بشكل متكرر لأنه لا يمكن إيجاد حل للعديد من هذه المعادلات بشكل مباشر أو يصعب حلها. يقدم هذا العمل طريقة لحل نوع من معادلة فريدهولم التفاضلية التكاملية غير الخطية (NFIDE) من النوع الثاني. تُستخدم قاعدة (Leibnitz) مع طريقة (Haar wavelet collocation) في هذه البحث لحل (NFIDE) عدديًا. استخدم بعض التقنيات لنقل المعادلة إلى نظام جبري من خلال مصفوفة تشغيلية. ان تحليل التقارب تم إثباته خلال هذا البحث وتم إعطاء تجارب عددية لكي توضح فعالية الطريقة المقترحة بناءً على برمجة MATLAB.