Hosoya Polynomials of Steiner Distance of an $m$-Cube and the Square of a Path

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ABSTRACT

The Hosoya polynomials of Steiner 3-distance of hypercube graphs $Q_m$, and of the square of a path, $P_t^2$, are obtained in this paper. The Steiner $n$-diameters of $Q_m$ and $P_t^2$ are also obtained.

1. Introduction.

We follow the terminology of [2,3]. For a connected graph $G = (V, E)$ of order $p$, the Steiner distance[4,5] of a non-empty subset $S \subseteq V(G)$, denoted by $d_G(S)$, or simply $d(S)$, is defined to be the size of the smallest connected subgraph $T(S)$ of $G$ that contains $S$; $T(S)$ is called a Steiner tree of $S$. If $|S|=2$, then $d(S)$ is the distance between the two vertices of $S$. For $2 \leq n \leq p$ and $|S|=n$, the Steiner distance of $S$ is called Steiner $n$-distance of $S$ in $G$. The Steiner $n$-diameter of $G$ (or the diameter of the Steiner $n$-distance), denoted by $diam_n^*G$ or $\delta_n^*(G)$, is defined as follows:

$$diam_n^*G = \max\{d_G(S) : S \subseteq V(G), |S| = n\}.$$

Remark 1.1. It is clear that

- (1) If $n \geq m$, then $diam_n^*G \geq diam_m^*G$.
- (2) If $S' \subseteq S$, then $d_G(S') \leq d_G(S)$.

The Steiner $n$-distance of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$, is the sum of the Steiner $n$-distances of all $n$-subsets containing $v$. The sum of Steiner $n$-distances of all $n$-subsets of $V(G)$ is denoted by $d_n(G)$ or $W_n^*(G)$. It is clear that
The graph invariant $W_n^*(G)$ is called Wiener index of the Steiner $n$-distance of the graph $G$.

**Definition 1.2** Let $C_n^*(G,k)$ be the number of $n$-subsets of distinct vertices of $G$ with Steiner $n$-distance $k$. The graph polynomial defined by

$$H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k)x^k,$$

where $\delta_n^*$ is the Steiner $n$-diameter of $G$; is called the Hosoya polynomial of Steiner $n$-distance of $G$.

It is clear that

$$W_n^*(G) = \sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k)$$

For $1 \leq n \leq p$, let $C_n^*(u,G,k)$ be the number of $n$-subsets $S$ of distinct vertices of $G$ containing $u$ with Steiner $n$-distance $k$. It is clear that

$$C_1^*(u,G,0) = 1.$$ Define

$$H_n^*(u,G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)x^k.$$ Obviously, for $2 \leq n \leq p$

$$H_n^*(G;x) = \frac{1}{n} \sum_{u \in V(G)} H_n^*(u,G;x).$$

Ali and Saeed \([1]\) were first whom studied this distance-based polynomial for Steiner $n$-distances, and established Hosoya polynomials of Steiner $n$-distance for some special graphs and graphs having some kind of regularity, and for Gutman’s compound graphs $G_1 \ast G_2$ and $G_1 : G_2$ in terms of Hosoya polynomials of $G_1$ and $G_2$.

In this paper, we obtain the Hosoya polynomial of Steiner 3-distance of $Q_m$ and $P_t^2$. Moreover, $diam_n^*Q_m$ and $diam_n^*P_t^2$ are determined.

2. **Hypercube Graphs** (m-Cube $Q_m$)

The Cartesian product \([3]\) of two connected disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph denoted by $G_1 \times G_2$ with
vertex set \( V \times V \) in which \((x_1, y_1)\) is joined to \((x_2, y_2)\) whenever \( \{x_1, x_2 \in E_1 \text{ and } y_1 = y_2\} \) or \( \{y_1, y_2 \in E_2 \text{ and } x_1 = x_2\} \).

If \( G_1 \) is a \((p_1, q_1)\)-graph and \( G_2 \) is a \((p_2, q_2)\)-graph, then \( G_1 \times G_2 \) is a \((p_1p_2, p_1q_2 + p_2q_1)\)-graph.

Now, the graph \( m\)-cube \( Q_m \) is defined recursively \([3]\) by \( Q_1 = K_2 \) and \( Q_m = Q_{m-1} \times K_2 \) for \( m \geq 2 \). Thus \( Q_m \) has \( 2^m \) vertices which may be labeled by the binary \( m\)-tuples \((s_1, s_2, \ldots, s_m)\) where each \( s_i \) is 0 or 1, for \( 1 \leq i \leq m \). Two vertices of \( Q_m \) are adjacent if their binary representations differ at exactly one place.

The diameter of \( Q_m \) is \( m[7] \), and \( Q_m \) is \( m\)-regular graph.

We next describe the Steiner \( n\)-diameter of the \( m\)-cube \( Q_m \).

**Proposition 2.1.** For \( m \geq 2 \) and \( n \geq 2^m - m + 1 \),
\[
\text{diam}_n^* Q_m = n - 1
\]

**Proof.** Since \( Q_m \) is \( m\)-connected \([3]\), so the removal of any \((m-1)\)-subset of vertices produces a connected subgraph of order \( 2^m - m + 1 \).
That is for any subset \( S \) of order \( n \geq 2^m - m + 1 \), the induced subgraph \( \langle S \rangle \) is connected, which implies that
\[
d(S) = n - 1
\]
This completes the proof. ■

**Proposition 2.2.** For \( m \geq 2 \) and \( 2 \leq n \leq 2^m - m \),
\[
\text{diam}_n^* Q_m \geq n
\]

**Proof.** We assume the contrary, that is we let \( \text{diam}_n^* Q_m < n \), then for any \( n\)-subset \( S \) of vertices of \( Q_m \), \( d(S) = n - 1 \). This means that the removal of any \( V - S \) subset of vertices produces a connected subgraph of \( Q_m \).
Thus, \( Q_m \) is \( (|V - S| + 1)\)-connected.
But \( |V - S| + 1 \geq 2^m - (2^m - m) + 1 = m + 1 \)
Contradicting the fact that \( Q_m \) is \( m\)-connected, so, we must have
\[
\text{diam}_n^* Q_m \geq n.
\]
Proposition 2.2 states that for \( 2 \leq n \leq 2^m - (m - 1) \), \( n \) is a lower bound for \( \text{diam}_n^* Q_m \). We can improve this bound in the next proposition.

**Proposition 2.3.** For \( 2 \leq n \leq 2^m - m \)
\[
\text{diam}_n^* Q_m \geq \max\{m, n\}
\]

**Proof.** It is clear that, this is true for \( m = 2 \) and \( m = 3 \).
It is known that \( \text{diam}_2^* Q_m = m \), and \( \max\{m, 2\} = m \geq 2 \),
So it is also true for \( n=2 \).

(a) If \( \max\{m,n\} = m \), that is \( m \geq n \), and if \( S \) contains \( u_0 = (0,0,\ldots,0) \) and \( u_m = (1,1,\ldots,1) \), then \( d(u_0,u_m) = m \) and \( d(S) \geq m \).

Therefore \( \text{diam}^*_n Q_m \geq m \).

(b) If \( \max\{m,n\} = n \), then by Proposition 2.2, \( \text{diam}^*_n Q_m \geq n \).

So, \( \text{diam}^*_n Q_m \geq \max\{m,n\} \) for \( 2 \leq n \leq 2^m - m \).

In the case of \( n=3 \), we have the following result.

**Proposition 2.4.** For \( m \geq 3 \)

\[ \text{diam}^*_3 Q_m = m . \]

**Proof.** The proof is by induction on \( m \).

It is clear that \( \text{diam}^*_3 Q_3 = 3 \), thus assume \( m \geq 3 \). Suppose that the result is true for \( m = k \geq 3 \), and consider \( m = k + 1 \).

Let \( S = \{u_1,u_2,u_3\} \) be any 3-subset of vertices of \( V(Q_{k+1}) \).

We know that

\[ Q_{k+1} = Q_k \times K_2. \]

If \( S \subseteq V(Q_k) \) or \( V(Q'_k) \), then by induction hypothesis \( d(S) \leq k \), where \( Q'_k \) is the second copy of \( Q_k \). (See Fig. 2.1).

![Fig. 2.1.](image)

Now, let \( u_1,u_2 \in V(Q_k) \) and \( u_3 \in V(Q'_k) \), and let \( u'_3 \) be a vertex in \( V(Q_k) \) adjacent to \( u_3 \) (see Fig.2.1), then

\[ d(\{u_1,u_2,u'_3\}) \leq k \]

Thus,

\[ \text{diam}^*_3 Q_{k+1} \leq k + 1 = m \]

By Proposition 2.3, \( \text{diam}^*_3 Q_m \geq m \), because \( 2 < 2^m - m \) for \( m \geq 3 \).

Thus,

\[ \text{diam}^*_3 Q_m = m . \]
We next investigate the Hosoya polynomial of Steiner 3-distance of \( Q_m \), which is obtained as a reduction formula in the following theorem.

**Theorem 2.5.** For \( m \geq 3 \),

\[
H_3^*(Q_m; x) = (2 + 6x)H_3^*(Q_{m-1}; x) + 4xH_2^*(Q_{m-1}; x),
\]

where

\[
H_2^*(Q_{m-1}; x) = 2^{m-2}(1 + x)^{m-1} - 2^{m-2}.
\]

**Proof.** Let \( S \) be a 3-subset of vertices of \( V(Q_m) \), and consider

\[
Q_m = Q_{m-1} \times K_2,
\]

assuming that \( Q_{m-1} \) and \( Q'_{m-1} \) are the two copies of the \((m-1)\)-cube in \( Q_m \).

We consider three cases for \( d_{Q_m} (S) \).

**Case I.** If \( S \subseteq V(Q_{m-1}) \) or \( V(Q'_{m-1}) \), then

\[
d_{Q_m} (S) = d_{Q_{m-1}} (S) = d_{Q'_{m-1}} (S).
\]

The Hosoya polynomial corresponding to all such \( S \) of this case is

\[
F_1(x) = 2H_3^*(Q_{m-1}; x).
\]

**Case II.** Let \( u,v,w \) be any 3 vertices of \( V(Q_{m-1}) \) and \( u',v',w' \) are the vertices of \( V(Q'_{m-1}) \) adjacent respectively to \( u,v,w \) as shown in Fig. 2.1 for \( k = m-1 \).

If \( S = \{u,v,w\}, \{u,v',w\}\{u',v,w\}, \{u',v',w\}\{u',v',w'\}\{u,v',w'\} \) or \( \{u,v',w'\} \) then

\[
d_{Q_m} (S) = 1 + d_{Q_{m-1}} (\{u,v,w\}).
\]

Thus, the Hosoya polynomial for all such six possibilities of \( S \) is

\[
F_2(x) = 6xH_3^*(Q_{m-1}; x)
\]

**Case III.** If \( S = \{u,u',v\}, \{u,u',w\}, \{u,u',v'\}, \{u,u',w'\}, \{u,u',v\} \) or \( \{u,u',w'\} \) then

\[
d_{Q_m} (S) = 1 + d_{Q_{m-1}} (S') = 1 + d_{Q_{m-1}} (S''),
\]

where

\[
S' = \{u,v\} \text{ or } \{u,w\} \text{ and } S'' = \{u',v'\} \text{ or } \{u',w'\} \text{ and } d_{Q_{m-1}} (S') \text{ and } d_{Q_{m-1}} (S'') \text{ denotes the ordinary distances of } S' \text{ and } S'' \text{ in } Q_{m-1} \text{ and } Q'_{m-1}, \text{ respectively}.
\]

Thus, the Hosoya polynomial for all such possibilities of \( S \) in this case is

\[
F_3(x) = 4xH_2^*(Q_{m-1}; x).
\]

Now, adding the polynomials \( F_1(x), F_2(x) \) and \( F_3(x) \) we obtain the required reduction formula.

Returning to the reduction formula obtained in Theorem 2.5, we find that \( H_3^*(Q_m; x) \) can be simplified as shown in the next corollary.
Corollary 2.6. For \( m \geq 3 \)
\[
H_3^*(Q_m; x) = 4x^2(2 + 6x)^{m-2} + 4x \sum_{k=1}^{m-2} (2 + 6x)^{k-1} H_2^*(Q_{m-k}; x)
\]

Proof. We know that
\[
H_3^*(Q_m; x) = (2 + 6x)H_3^*(Q_{m-1}; x) + 4xH_2^*(Q_{m-1}; x)
\]
\[
= (2 + 6x)[(2 + 6x)H_3^*(Q_{m-2}; x) + 4xH_2^*(Q_{m-2}; x)] + 4xH_2^*(Q_{m-1}; x)
\]
\[
= (2 + 6x)^2 H_3^*(Q_{m-2}; x) + 4x[(2 + 6x)H_2^*(Q_{m-2}; x) + H_2^*(Q_{m-1}; x)]
\]
\[
= (2 + 6x)^{m-2} H_3^*(Q_2; x) + 4x[(2 + 6x)^{m-3} H_2(Q_{m-(m-2)}; x) + ... + (2 + 6x)^{m-4} H_2(Q_{m-(m-3)}; x) + H_2(Q_{m-1}; x)]
\]
It is obvious that \( H_3(Q_2; x) = 4x^2 \)
Hence
\[
H_3^*(Q_m; x) = 4x^2(2 + 6x)^{m-2} + 4x \sum_{r=2}^{m-1} (2 + 6x)^{m-1-r} H_2(Q_r; x).
\]

Next corollary computes the Wiener index of Steiner 3-distance of \( Q_m \).

Corollary 2.7. For \( m \geq 3 \)
\[
W_3^*(Q_m) = 8^{m-2}(3m + 2) + 2^{m-4} \sum_{k=1}^{m-2} 4^k \{2^{m-k+1}(m - k) + (2^{m-k} - 1)(3k + 1)\}.
\]

3. The Square of a Path (\( P_t^2 \))

The \( n^{th} \) power \( G^n \) of a connected graph \( G \) has vertex set \( V(G) \) and for each distinct vertices \( u,v \) of \( G^n, uv \in E(G^n) \) whenever
\[
1 \leq d_G(u,v) \leq n.
\]
It is clear that, if \( diamG = n \) then \( G^n \) is a complete graph.

In [7], W. A. M. Saeed proved that
\[
diamG^n = \left\lceil \frac{diamG}{n} \right\rceil.
\]

In this section, we consider the square \( P_t^2 \) of a path \( P_t \), with respect to Steiner distance. First, we find the Steiner \( n \)-diameter.

Proposition 3.1. For even \( t \geq 4 \), and for \( 2 \leq n \leq t \), the Steiner \( n \)-diameter of \( P_t^2 \) is
\[
t \geq \frac{t}{2} - 1 + \left\lceil \frac{n}{2} \right\rceil.
\]

Proof. The graph \( P_t^2 \) is shown in Fig.3.1.
Let $P_t = u_1, u_2, \ldots, u_t$, then
\[
V(P_t^2) = V(P_t) = \{u_1, u_2, \ldots, u_t\}.
\]
If $S$ is an $n$-subset of vertices of $V(P_t^2)$ such that $d(S)$ is maximum, then $S$ must contain the two vertices $u_1$ and $u_t$, the other vertices of $S$ must be the first $n-2$ vertices from the sequence (See Fig. 3.1).

Therefore $S$ contains $\left\lfloor \frac{n-2}{2} \right\rfloor$ vertices from one of the sets $A = \{u_2, u_4, \ldots, u_{t-2}\}$, $B = \{u_3, u_5, \ldots, u_{t-1}\}$ and contains $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from the other set. If $S$ contains $\left\lfloor \frac{n-2}{2} \right\rfloor$ vertices from $A$, then $T(S)$ must contain the $u_1 - u_t$ path $u_1, u_2, u_4, \ldots, u_{t-2}, u_t$, and so $S$ will contain the $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from $B$, and the size of $T(S)$ will be $\frac{t + \left\lfloor \frac{n-2}{2} \right\rfloor}{2}$. But if $S$ contains $\left\lceil \frac{n-2}{2} \right\rceil$ vertices from $B$, then $T(S)$ must contain the $u_1 - u_t$ path $u_1, u_3, u_5, \ldots, u_{t-1}, u_t$, and the size of $T(S)$ will also be $\frac{t + \left\lceil \frac{n-2}{2} \right\rceil}{2}$.

Hence, the proof of the proposition.

**Proposition 3.2.** For odd $t \geq 3$, $2 \leq n \leq t$, the Steiner $n$-diameter of $P_t^2$ is $\frac{t-3}{2} + \left\lceil \frac{n}{2} \right\rceil$.

**Proof:** The proof is similar to that of Proposition 3.1. It is clear that there is exactly one shortest $u_1 - u_t$ path in $P_t^2$ whose length is $\frac{t-1}{2}$, namely...
The other \((n - 2)\) vertices of the \(n\)-subset \(S\) are the first \(n - 2\) from the sequence \(u_2, u_3, u_4, \ldots, u_{t-1}\). Therefore \(S\) will contain the first \(\left\lceil \frac{n - 2}{2} \right\rceil\) vertices from \(\{u_2, u_4, \ldots, u_{t-1}\}\).

Thus \(S\) of maximum Steiner \(n\)-distance has
\[
d(S) = \frac{t - 1}{2} + \left\lceil \frac{n - 2}{2} \right\rceil = \frac{t - 3}{2} + \left\lceil \frac{n}{2} \right\rceil.
\]

Next, we find Hosoya polynomial of the Steiner \(3\)-distance of the square of a path \(P_t\).

**Theorem 3.3.** Let \(t = 2s \geq 6\) be an even positive integer, then
\[
H_3^*(P_t^2; x) = H_3^*(P_{t-2}^2; x) + F_s(x)
\]
where
\[
F_s(x) = 2x^2 + 2x^s + \sum_{j=2}^{s-1} [4(x + 1)j - 2x - 2]x^j
\]

**Proof.** The graph \(P_t^2\) is shown in Fig.3.1; its vertices are relabeled as shown in Fig.3.2 in order to simplify the derivation of \(F_s(x)\).

![Diagram of \(P_t^2\)](image)

Next, we find Hosoya polynomial of the Steiner \(3\)-distance of the square of a path \(P_t\).

Let \(P_{t-2}^2\) be obtained from \(P_t^2\) by deleting the two vertices \(v_s, v'_s\). Then
\[
H_3^*(P_t^2; x) = H_3^*(P_{t-2}^2; x) + F_s(x)
\]
where
\[
F_s(x) = \sum_S x^{d(S)},
\]
in which \(|S| = 3\), \(S \cap \{v_s, v'_s\} \neq \emptyset\) and \(S \cap V(P_{t-2}^2) \neq \emptyset\).

To find \(F_s(x)\) we consider several cases for \(S\).

(1) If \(S = \{v_s, v'_s, w\}\), \(w \in V(P_{t-2}^2)\), then
\[
d(S) = s + 1 - i, \quad \text{when } w = v_i \text{ or } v'_i, \quad 1 \leq i \leq s - 1.
\]
Thus, the polynomial corresponding to all such \(S\)’s of this case is
\[ f_1(x) = 2 \sum_{i=1}^{s-1} x^{s+1-i} = 2 \sum_{j=2}^{s} x^j. \]

(2) If \( S = \{v_x, v_i, v_j\}, \ 1 \leq i < j \leq s - 1, \) then
\[ d(S) = s - i. \]

It is clear that for each value of \( i \) there are \((s - i - 1)\) values of \( j \). Thus the corresponding polynomial is \( \sum_{i=1}^{s-1} (s - i - 1)x^{s-i} \).

The same polynomial is obtained if \( S = \{u'_s, u'_i, u'_j\} \).

Therefore, for such 3-subsets \( S \) we get
\[ f_2(x) = 2 \sum_{i=1}^{s-2} (s - i - 1)x^{s-i} = 2 \sum_{j=2}^{s-1} (j - 1)x^j. \]

(3) If \( S = \{v_x, v_i, v'_j\} \) or \( \{v'_x, v_i, v'_j\} \), then
\[ d(S) = s - i + 1, \ 1 \leq i \leq s - 1. \]

Thus, the corresponding polynomial is
\[ f_3(S) = 2 \sum_{i=1}^{s-1} x^{s-i+1} = 2 \sum_{j=1}^{s} x^{j+1}. \]

(4) If \( S = \{v_x, v'_i, v'_j\}, \ 1 \leq i < j \leq s - 1, \) then
\[ d(S) = s + 1 - i. \]

Similarly, if \( S = \{v_x, v'_i, v_j\}, \ 1 \leq i < j \leq s - 1, \) then
\[ d(S) = s - i. \]

Thus, the corresponding polynomial is
\[ f_4(x) = \sum_{i=1}^{s-2} (s - i - 1)x^{s+1-i} + \sum_{i=1}^{s-2} (s - i - 1)x^{i-1} = \sum_{j=2}^{s} (j - 1)(x + 1)x^j. \]

(5) If \( S = \{v'_x, v'_i, v'_j\}, \ 1 \leq i < j \leq s - 1, \) then
\[ d(S) = s - i + 1, \]
and there are \((s - i - 1)\) values for \( j \).

Similarly, if \( S = \{v'_x, v'_i, v_j\} \) then \( d(S) = s - i + 1 \) for \( 1 \leq i < j \leq s - 1 \).

Thus, the polynomial corresponding to all these 3-subsets is
\[ f_5(x) = 2 \sum_{i=1}^{s-2} (s - i - 1)x^{s-i+1} = 2 \sum_{j=2}^{s} (j - 1)x^{j+1}. \]

(6) If \( S = \{v_x, v'_i, v'_j\}, \ 1 \leq i < j \leq s - 1, \) then
\[ d(S) = s - i. \]

The corresponding polynomial is
Similarly, if $S = \{v'_i, v_i, v_j\}$, $1 \leq i < j \leq s - 1$, then $d(S) = s - i + 1$.

The corresponding polynomial for such $S$ is

$$
\sum_{i=1}^{s-2} (s - i - 1)x^{s-i}.
$$

Thus, the distance polynomial for all these 3-subsets $S$ in this case is

$$
f_6(x) = \sum_{i=1}^{s-2} (s - i - 1)x^{s-i} + \sum_{i=1}^{s-2} (s - i - 1)x^{s-i+1}
$$

$$
= \sum_{j=2}^{s-1} (j - 1)(x + 1)x^j
$$

These are all possibilities of $S$. Therefore

$$
F_s(x) = \sum_{r=1}^{6} f_r(x)
$$

$$
= 2x^2 + 2x^s + 2\sum_{j=2}^{s-1} x^j + 2\sum_{j=2}^{s-1} (j - 1)x^j + 2\sum_{j=2}^{s-1} x^{j+1}
$$

$$
+ 2\sum_{j=2}^{s-1} (j - 1)(x + 1)x^j + 2\sum_{j=2}^{s-1} (j - 1)x^{j+1}.
$$

Simplifying the above summations we get the reduction formula given in the theorem.

The Wiener index of the Steiner 3-distance of $P_t^2$ for even $t$ is given in the next corollary.

**Corollary 3.4.** For $t = 2s \geq 4$,

$$
W^*_3(P_t^2) = W^*_3(P^{2}_{t-2}) + \frac{4}{3} s(s - 1)(2s - 1).
$$

We now consider the square of a path $P_t$ of odd order $t = 2s + 1$.

The next theorem gives us a reduction formula of $H^*_3(P_t^2; x)$.

**Theorem 3.5.** For $t = 2s \geq 7$, we have

$$
H^*_3(P_t^2; x) = H^*_3(P_{t-1}^2; x) + F_s(x),
$$

where

$$
F_s(x) = x^2 + \sum_{j=1}^{s-1} [(x + 3)j + x]x^{j+1}.
$$

**Proof:** The graph $P_t^2$ is shown in Fig. 3.3 where the vertices are labeled as that of Fig. 3.2.
Fig. 3.3. \( P_t^2, \) odd \( t \)

\( P_{t-1}^2 \) is obtained from \( P_t^2 \) by removing vertex \( v_{s+1} \). Thus

\[ H_3^*(P_t^2; x) = H_3^*(P_{t-1}^2; x) + F_s(x), \]

where

\[ F_s(x) = \sum_S x^{d(S)}, \]

in which the summation is taken over all 3-subsets \( S \)

\[ S = \{v_{s+1}, u_i, u_j\} \text{ for all } u_i, u_j \in V(P_{t-1}^2). \]

We consider the following 5 cases.

1. If \( S = \{v_{s+1}, v_i, v_j\}, 1 \leq i < j \leq s \), then

\[ d(S) = s + 1 - i. \]

The number of values of \( j \) is \((s-i)\) for each values of \( i \). Thus, the polynomial corresponding to such 3-subsets \( S \) of this case is

\[ f_1(x) = \sum_{i=1}^{s-1} (s-i)x^{s+1-i} = \sum_{j=1}^{s-1} jx^{j+1}. \]

2. If \( S = \{v_{s+1}, v_i, v'_i\}, 1 \leq i \leq s \), then

\[ d(S) = s + 2 - i. \]

Therefore the corresponding polynomial is

\[ f_2(x) = \sum_{i=1}^{s} x^{s+2-i} = x^2 + x^{s-1} \sum_{j=1}^{s} x^j. \]

3. If \( S = \{v_{s+1}, v'_i, v'_j\}, 1 \leq i < j \leq s \), then

\[ d(S) = s - i + 1, \]

and for each value of \( i \) there are \((s-i)\) values for \( j \). Thus, the corresponding polynomial for such case of \( S \) is

\[ f_3(x) = \sum_{i=1}^{s-1} (s-i)x^{s-i+1} = \sum_{j=1}^{s-1} jx^{j+1}. \]

4. If \( S = \{v_{s+1}, v_i, v'_j\}, 1 \leq i < j \leq s \), then

\[ d(S) = s + 2 - i, \]

and for each value of \( i \) there are \((s-i)\) values for \( j \). Thus, the polynomial corresponding to all 3-subsets \( S \) of this case is
\[ f_4(x) = \sum_{i=1}^{s-1} (s-i)x^{s+2-i} = x^2 \sum_{j=1}^{s-1} jx^j. \]

(5) Finally, If \( S = \{ v_{s+1}, v_i', v_j \} \), \( 1 \leq i < j \leq s \), then \( d(S) = s + 1 - i \), and there are \((s-i)\) values for \( j \) for each value of \( i \). Therefore, the corresponding polynomial is
\[ f_s(x) = \sum_{i=1}^{s-1} (s-i)x^{s+1-i} = \sum_{j=1}^{s-1} jx^{j+1}. \]

Thus,
\[ F_s(x) = \sum_{r=1}^{5} f_r(x) = \sum_{j=1}^{s-1} (jx + x^2 + jx + x^2j + jx)x^j + x^2 \]
\[ = x^2 + \sum_{j=1}^{s-1} [(x + 3)j + x]x^{j+1}. \]

The next corollary gives us the Wiener index of the Steiner 3-distance of \( P_t^2 \) for odd \( t \).

**Corollary 3.6.** For odd \( t = 2s + 1 \), \( s \geq 2 \), the Wiener index of \( P_t^2 \) is
\[ W_3^*(P_t^2) = W_3^*(P_{t-1}^2) + \frac{1}{3}(s-1)(4s^2 + 7s + 6) + 2. \]

**References**