Solvability Conditions For A system Of Nonhomogenuous Differential Equations Of The First Order

AKRAM.H.MAAMOOD
LAMYAA.H.SAADOON & RAID.S.KARYAKOS
Department Of Mathematics/College Of Education
University of Mosul

Accepted 17/07/2007
Received 24/04/2007

Abstract

In this paper we study the solvability conditions for a system of Nonhomogenous ordinary differential equations of the first order with boundary conditions by using the method which is given by [3,4].

1-Introduction :
The author [2] investigated the solvability conditions for certain eigenvalue problems by using perturbation method.

Also [1] investigated the solvability conditions of boundary value problem for partial differential equations of the 2nd order by using the same method which is given by [3].

Our work is to find the solvability conditions for a system of the first order of differential equations by using the perturbation method which is given by [3].

We consider the following system of differential equations

\[ \frac{dy}{dx} - A(x)y = f(x) \]  
(1.1)

with boundary conditions

\[ y_i(a) = \beta_i \quad \text{for} \quad i=1,2,\ldots,m \]  
(1.2)

\[ y_i(b) = \beta_i \quad \text{for} \quad i=m+1,m+2,\ldots,n \]

where \( y \) and \( f \) are column vectors with \( n \) components, \( A \) is an \( n \times n \) matrix, and \( \beta_i \) are constants.

2-Main Results:
In this section we formulate the main results by depending on the following theorem:
Theorem:
The necessary conditions for solvability of a system of ordinary differential equation (1.1) with conditions (1.2) is:

\[ \beta_{m+1} Z_{m+1}(b) + \beta_{m+2} Z_{m+2}(b) + \ldots + \beta_n Z_n(b) - \beta_1 Z_1(a) - \beta_2 Z_2(a) - \ldots - \beta_m Z_m(a) = \int_a^b Z^T f d(x) \]  

(2.1)

Proof:
We consider the solvability conditions for the problem

\[ \frac{dy}{dx} - A(x)y = f(x) \]  

(2.2)

Is

\[ Y_1(a) = \beta_i \quad \text{for} \quad i=1,2,\ldots,m \]  

(2.3)

\[ Y_i(b) = \beta_i \quad \text{for} \quad i=m+1,m+2,\ldots,n \]

where \( y \) and \( f \) are column vectors with \( n \) components, \( A \) is an \( n \times n \) matrix, and \( \beta_i \) are constants.

We assume that the corresponding homogenous problem of (2.2,2.3) has a nontrivial solution, so that the nonhomogenous problem will have a solution only if solvability conditions are satisfied.

To determine the solvability conditions we multiply (2.2) from the left with \( Z^T \) where \( Z^T \) is the transpose of the adjoint column vector \( z \) with \( n \) components.

Thus, we have:

\[ Z^T \frac{dy}{dx} - Z^T Ay = Z^T f \]

which, upon integration from \( x = a \) to \( x = b \) gives:

\[ \int_a^b Z^T \frac{dy}{dx} dx - \int_a^b Z^T Ay dx = \int_a^b Z^T f dx \]  

(2.4)

Integration by parts the first integral on the left-hand side of (2.4) we find:

\[ Z^T y \bigg|_a^b - \int_a^b \frac{d}{dx} Z^T y dx - \int_a^b Z^T Ay dx = \int_a^b Z^T f dx \]

or
The adjoint equations are defined by setting the coefficient of $y$ in the integrand in (2.5) equal to zero and obtaining
\[
\frac{d}{dx} z^T + z^T A = 0
\]

Taking the transpose, we have:
\[
\left( \frac{d}{dx} z^T \right)^T + (z^T A)^T = 0
\]

or
\[
\frac{dz}{dx} + A^T z = 0
\]  

(2.6)

Comparing (2.6) and (2.2), we conclude that the differential equations are self-adjoint if $A=-A^T$.

To determine the boundary conditions on $z$, we consider the corresponding homogenous problem by setting $f=0$ in (2.5), we obtain:

\[
\left. z^T y \right|_a^b - \int_a^b (\frac{d}{dx} z^T + z^T A) y dx = \int_a^b z^T f dy
\]  

(2.5)

Putting $\beta_1 = 0$ in (2.3) and substituting the result into (2.8), we have:

\[
Z_i (b) y_1 (b) + Z_2 (b) y_2 (b) + \ldots + Z_m (b) y_m (b) - \sum_{i=1}^{m+1} a_i y_{m+1} (a) - 
- \sum_{i=m+2}^{n} a_i y_n (a) = 0
\]  

(2.9)

We define the adjoint boundary conditions such that each of the coefficient of the $y_i (b)$ for $i=1,2,3,\ldots,m$ and the $y_i (a)$ for $i=m+1, m+2, \ldots, n$ equal to zero, so that the result is:

\[
Z_i (a) = 0 \quad \text{for} \quad i=m+1, m+2, \ldots, n
\]

(2.10)

\[
Z_i (b) = 0 \quad \text{for} \quad i=1,2, \ldots, m
\]
Returning to the nonhomogenous problem, we substituting (2.3), (2.6) and (2.10) into (2.5) we find:

\[ \beta_{m+1} Z_{m+1}(b) + \beta_{m+2} Z_{m+2}(b) + \ldots + \beta_n Z_n(b) - \beta_1 Z_1(a) - \beta_2 Z_2(a) - \ldots - \]

\[ - \beta_m Z_m(a) = \int_a^b Z^T f \, dx \]

As the desired solvability conditions.

**EXAMPLE:**

To illustrate the procedure, we will specify the following system differential equations:

\[
\begin{pmatrix}
  y_1' \\
  y_2'
\end{pmatrix} - \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  0 \\
  x
\end{pmatrix}
\]

with the boundary conditions

\[ y_1(0) = 0.25 \]

\[ y_2(\pi/2) = 0.75 \]

\[ \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix} y_1' \\
  y_2'
\end{pmatrix} - \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix} 0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix}
  0 \\
  0
\end{pmatrix} \]

\[ \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix} y_1' \\
  y_2'
\end{pmatrix} \bigg|_0^{\pi/2} - \int_0^{\pi/2} \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix} 0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} \, dx + \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{pmatrix} 0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  y_1' \\
  y_2'
\end{pmatrix} \bigg|_0^{\pi/2} \]

\[ \begin{bmatrix} z_1' \\
  z_2'
\end{bmatrix} + \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix} = \begin{pmatrix} 0 \\
  0
\end{pmatrix} \]
\[ Z_1 Y_1 + Z_2 Y_2 \bigg|_{\pi/2}^{0} = 0 \]
\[ Z_1 (\pi/2) Y_1 (\pi/2) - Z_2 (0) Y_2 (0) = 0 \]
\[ Z_1 (\pi/2) = 0 \]
\[ Z_2 (0) = 0 \]
\[ 0.75 Z_2 (\pi/2) - 0.25 Z_1 (0) = \int_{0}^{\pi/2} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} dx \]
\[ 0.75 Z_2 (\pi/2) - 0.25 Z_1 (0) = \int_{0}^{\pi/2} x Z_2 (x) dx \]
\[ Z_1 (x) = \cos(x) \]
\[ Z_2 (x) = -\sin(x) \]
\[ Z_1 (0) = 1 \]
\[ Z_2 (\pi/2) = -1 \]

If we compare with (2.1), the desired solvability conditions is
\[ -0.75 - 0.25 = -\int_{0}^{\pi/2} x \sin(x) dx \]
\[ -1 = -1 \]

References:
1- محمود ، اكرم حسان وجاسم ،اسراء  عبد العالي ، شروط الحل لمسلسل حل محدودة خاصة

بطريقة التشويش ، مجلة التربية والعلم ، مجلد (18) ، 2006