Some basic properties of idempotent matrices

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Abstract

In this paper we give some properties of the zero commut idempotent matrices ,and give some properties of non-singular matrices .
Some basic properties of idempotent matrices.

1. Introduction

Throughout, this paper all matrices considered are square and commutative unless otherwise stated.

In [2], Koliha, Racocevic and Straskraba present new results on the invertibility of the sum of projectors, new relations between the non-singularity of the difference and the sum of projectors, and present a simple proof of the invertibility of n×n matrix A exists by showing that N(A)={0}. In this paper we present some basic properties of an idempotent matrices and relations between the range and the null spaces and give some results on the non-singularity of the difference and sum of idempotent matrices. We recall that:

1) A matrix A is said to be idempotent if \( A^2 = A \).
2) A two matrices A and B are said to be zero commut if \( AB = BA = 0 \).
3) A null space of a matrix A is the set of all solutions to the equation \( A \mathbf{x} = 0 \), \( \text{Null}(A) = \{ \mathbf{x} \in \mathbb{R}^n : A \mathbf{x} = 0 \} \), and we denote that N(A).
4) A range space of a matrix A is the set of all solutions to the equation \( A \mathbf{x} = \mathbf{v} \), and we denote that R(A).

2. Idempotent matrices

In this section we give some basic properties of the idempotent matrices.

Lemma 2.1:

If A is idempotent matrix, then (I-A) is idempotent.

Proof:

Trivial.

Proposition 2.2 [2]:

If A is a matrix, then \( \text{N}(A) = \text{R}(I - A) \), also \( \text{N}(I - A) = \text{R}(A) \).

Proposition 2.3:

If A is a matrix, then \( \text{R}(A) \cap \text{N}(A) = \{0\} \).

Proof:

Let \( \mathbf{x} \in \text{R}(A) \cap \text{N}(A) \), then \( \mathbf{x} \in \text{R}(A) \) and \( \mathbf{x} \in \text{N}(A) \).

Then \( \mathbf{x} = A \mathbf{x} \) and \( A \mathbf{x} = 0 \), so, \( \mathbf{x} = 0 \).

Hence \( \text{R}(A) \cap \text{N}(A) = \{0\} \).

Proposition 2.4:

If A and B be are idempotent matrices, then \( \text{R}(A) \cap \text{R}(B(I - A)) = \{0\} \).

Proof:

Let \( \mathbf{y} \in \text{R}(A) \cap \text{R}(B(I - A)) \), then \( \mathbf{y} = A \mathbf{y} \) and \( \mathbf{y} = B(I - A) \mathbf{y} \).
So \( A\bar{y} = AB(I-A)\bar{y} = AB\bar{y} - ABA\bar{y} = 0 \).

But \( A\bar{y} = \bar{y} \), so \( \bar{y} = 0 \).

Hence \( R(A) \cap R[B(I-A)] = \{0\} \).

**Proposition 2.5:**
If \( A \) and \( B \) be are matrices, with \( AB = A \) and \( BA = B \). Then \( N(A) = N(B) \).

**Proof:**

Let \( \bar{x} \in N(A) \), then \( A\bar{x} = 0 \), so \( BA \bar{x} = 0 \), but \( BA = B \). Therefore \( B\bar{x} = 0 \) and \( \bar{x} \in N(B) \). Hence \( N(A) \subseteq N(B) \).

Similarly \( N(B) \subseteq N(A) \).

Therefore \( N(A) = N(B) \).

**Proposition 2.6:**
If \( A \) and \( B \) be are idempotent matrices, then \( R(AB) = R(A) \cap R(B) \).

**Proof:**

Let \( x\bar{v} \in R(AB) \), then \( x\bar{v} = ABx\bar{v} \).

So \( (I-A)x\bar{v} = (I-A)ABx\bar{v} = 0 \).

Hence \( x\bar{v} \in N(I-A) = R(A) \) [by Proposition 2.2].

Similarly \( x\bar{v} \in R(B) \), so \( \bar{x} \in R(A) \cap R(B) \).

Therefore \( R(AB) \subseteq R(A) \cap R(B) \).

Now, let \( \bar{y} \in R(A) \cap R(B) \), then \( \bar{y} = A\bar{y} \) and \( \bar{y} = B\bar{y} \).

So \( A\bar{y} = B\bar{y} \). That is \( A\bar{y} = AB\bar{y} \),

but \( \bar{y} = A\bar{y} \), therefore \( \bar{y} = AB\bar{y} \in R(AB) \).

So \( R(A) \cap R(B) \subseteq R(AB) \).

Hence \( R(AB) = R(A) \cap R(B) \).

**Proposition 2.7:**
If \( A \) and \( B \) be are idempotent matrices, and \( R(A) \cap R(B) = \{0\} \), then \( N(A - B) = N(A) \cap N(B) \).

**Proof:**

Let \( \bar{x} \in N(A-B) \), then \( (A-B)\bar{x} = 0 \) and \( A\bar{x} = B\bar{x} \).

So \( A\bar{x} = A^2\bar{x} = AB\bar{x} = BA\bar{x} \).

Now, from \( A\bar{x} = BA\bar{x} \) we get \( (B-I)A\bar{x} = 0 \), so \( A\bar{x} \in N(B-I) = R(B) \) [by Proposition 2.2], but \( A\bar{x} = B\bar{x} \), so \( B\bar{x} \in R(B) \).

Similarly from \( B\bar{x} = AB\bar{x} \) we get \( A\bar{x}, B\bar{x} \in R(A) \).

So \( A\bar{x}, B\bar{x} \in R(A) \cap R(B) = \{0\} \), therefore \( \bar{x} \in N(A) \).

Also, \( \bar{x} \in N(B) \), so \( \bar{x} \in N(A) \cap N(B) \).

Hence \( N(A-B) \subseteq N(A) \cap N(B) \).

Now, let \( \bar{y} \in N(A) \cap N(B) \), then \( A\bar{y} = 0 \) and \( B\bar{y} = 0 \).

So \( (A-B)\bar{y} = 0 \), therefore \( \bar{y} \in N(A-B) \).

So \( N(A) \cap N(B) \subseteq N(A-B) \).

Hence \( N(A-B) = N(A) \cap N(B) \).
**Proposition 2.8**:  
If $A$ and $B$ be are idempotent matrices, then $R[B(I-A)]=N(A) \cap R(B)$.

**Proof:**  
Let $\bar{x} \in R[B(I-A)]$, then $\bar{x} = B(I-A)\bar{x}$.
So $A\bar{x} = AB(I-A)\bar{x} = AB\bar{x} - ABA\bar{x} = 0$.
So $\bar{x} \in N(A)$.
Also, $(I-B)\bar{x} = (I-B)(I-A)\bar{x} = 0$, so $\bar{x} = B\bar{x}$, therefore $\bar{x} \in R(B)$.
So $\bar{x} \in N(A) \cap R(B)$.
Hence $R[B(I-A)] \subseteq N(A) \cap R(B)$.
Now, let $\bar{y} \in N(A) \cap R(B)$ then $A\bar{y} = 0$ and $\bar{y} = B\bar{y}$.
So $A\bar{y} = AB\bar{y} = 0$.
Now, $B(I-A)\bar{y} = B(I-A)B\bar{y} = B\bar{y} - BAB\bar{y}$.
Therefore $B(I-A)\bar{y} = B\bar{y} = \bar{y}$, so $\bar{y} \in R[B(I-A)]$.
Hence $N(A) \cap R(B) \subseteq R[B(I-A)]$.
Therefore $R[B(I-A)] = N(A) \cap R(B)$.

**Proposition 2.9**:  
If $A$ and $B$ be are idempotent matrices, then $R(A) = R(B)$ if and only if $N(A) = N(B)$.

**Proof:**  
Let $R(A) = R(B)$, then $A\bar{x} = B\bar{x}$ and let $\bar{x} \in N(A)$, then $A\bar{x} = 0$, so $B\bar{x} = 0$.
Then $\bar{x} \in N(B)$, therefore $N(A) \subseteq N(B)$.
Similarly $N(B) \subseteq N(A)$, so $N(A) = N(B)$.
Now, let $N(A) = N(B)$ and let $\bar{y} \in R(A)$, then $\bar{y} = A\bar{y}$.
So $B\bar{y} = BA\bar{y} = AB\bar{y}$.
From $B(I-B)\bar{y} = 0$, we get $(I-B)\bar{y} \in N(B) = N(A)$ and $A(I-B)\bar{y} = 0$.
So $A\bar{y} = AB\bar{y}$, but $\bar{y} = A\bar{y}$ and $B\bar{y} = AB\bar{y}$.
Therefore $\bar{y} = B\bar{y}$ and $\bar{y} \in R(B)$, so $R(A) \subseteq R(B)$.
Similarly $R(B) \subseteq R(A)$.
Hence $R(A) = R(B)$.

**Proposition 2.10**:  
If $A$ and $B$ be are idempotent matrices, then $N(A) \cap N(B) = N(A+B-AB)$.

**Proof:**  
Let $\bar{x} \in N(A) \cap N(B)$, then $A\bar{x} = 0$ and $B\bar{x} = 0$, so $AB\bar{x} = 0$.
Therefore $(A+B-AB)\bar{x} = 0$ and $\bar{x} \in N(A+B-AB)$.
So $N(A) \cap N(B) \subseteq N(A+B-AB)$.
Now, let $\bar{y} \in N(A+B-AB)$, then $(A+B-AB)\bar{y} = 0$, so $A\bar{y} + B\bar{y} = AB\bar{y}$.
Hence $A^2 \bar{y} + AB\bar{y} = A^2 B\bar{y}$. So $A\bar{y} = 0$ and $\bar{y} \in N(A)$. 

Similarly \( B\tilde{y} = 0 \) and \( \tilde{y} \in \text{N}(B) \).
So \( \tilde{y} \in \text{N}(A) \cap \text{N}(B) \).
Hence \( \text{N}(A+B-AB) \subseteq \text{N}(A) \cap \text{N}(B) \).
Therefore \( \text{N}(A) \cap \text{N}(B) = \text{N}(A+B-AB) \).  ■

3. Basic properties of zero commut idempotent matrices.
In this section we present some basic properties of the zero commut idempotent matrices.

**Proposition 3.1:**
If \( A \) and \( B \) be are zero commut idempotent matrices, then
\( \text{N}(A-B) = \text{N}(A) \cap \text{N}(B) \).

**Proof:**
Let \( \tilde{x} \in \text{N}(A-B) \), then \( (A-B)\tilde{x} = 0 \) and \( A\tilde{x} = B\tilde{x} \).
So \( A\tilde{x} = AB\tilde{x} = 0 \).
Hence \( \tilde{x} \in \text{N}(A) \).
Similarly \( \tilde{x} \in \text{N}(B) \), so \( \tilde{x} \in \text{N}(A) \cap \text{N}(B) \).
Hence \( \text{N}(A-B) \subseteq \text{N}(A) \cap \text{N}(B) \).
Now, let \( \tilde{y} \in \text{N}(A) \cap \text{N}(B) \), then \( A\tilde{y} = 0 \) and \( B\tilde{y} = 0 \), so \( (A-B)\tilde{y} = 0 \).
Therefore \( \tilde{y} \in \text{N}(A-B) \), so \( \text{N}(A) \cap \text{N}(B) \subseteq \text{N}(A-B) \).
Hence \( \text{N}(A-B) = \text{N}(A) \cap \text{N}(B) \).  ■

**Proposition 3.2:**
If \( A \) and \( B \) be are zero commut idempotent matrices, then
\( \text{N}(A+B) = \text{N}(A) \cap \text{N}(B) \).

**Proof:**
Similar to the proof of proposition 3.1

**Corollary 3.3:**
If \( A \) and \( B \) be are zero commut idempotent matrices, then
\( \text{N}(A+B) = \text{N}(A-B) \).

**Proposition 3.4:**
If \( A \) and \( B \) be are zero commut idempotent matrices, then
\( \text{R}(A+B) = \text{R}(A)+\text{R}(B) \).

**Proof:**
Let \( \tilde{y} = \tilde{y}_1 + \tilde{y}_2 \in \text{R}(A) + \text{R}(B) \),
where \( \tilde{y}_1 \in \text{R}(A) \) and \( \tilde{y}_2 \in \text{R}(B) \).
So \( (A+B)(\tilde{y}_1 + \tilde{y}_2) = A\tilde{y}_1 + A\tilde{y}_2 + B\tilde{y}_1 + B\tilde{y}_2 \)
\[ = \tilde{y}_1 + A\tilde{y}_2 + B\tilde{y}_1 - 0 \]
\[ = \tilde{y}_1 + A\tilde{y}_2 + B\tilde{y}_1 - AB(\tilde{y}_1 + \tilde{y}_2) \]
Some basic properties of idempotent matrices.

\[
= y_1 + y_2 + A(y_2 - By_2) + B(y_1 - Ay_1)
= y_1 + y_2 = \bar{y}.
\]
Therefore \( \bar{y} = y_1 + y_2 \in R(A+B) \), so \( R(A) + R(B) \subseteq R(A+B) \).

Now, let \( \bar{x} \in R(A+B) \), then \( \bar{x} = (A+B)\bar{x} = A\bar{x} + B\bar{x} \).
Let \( \bar{x}_1 = A\bar{x} \), then \( A\bar{x}_1 = A\bar{x} = \bar{x}_1 \), so \( \bar{x}_1 \in R(A) \).
Also, let \( \bar{x}_2 = B\bar{x} \), then \( B\bar{x}_2 = B\bar{x} = \bar{x}_2 \), so \( \bar{x}_2 \in R(B) \).
Therefore \( \bar{x} = \bar{x}_1 + \bar{x}_2 \in R(A) + R(B) \), so \( R(A+B) \subseteq R(A) + R(B) \).
Hence \( R(A+B) = R(A) + R(B) \) .

**Corollary 3.5:**
If \( A \) is an idempotent matrix, then \( R^n = R(A) + N(A) \).

**Proof:**
Since \( A(I-A) = (I-A)A = 0 \).
Then \( A \) and \( I-A \) are zero commut idempotent matrices.
Now, from [Proposition 3.4] we get \( R(A+I-A) = R(A) + R(I-A) \).
Hence \( R(I) = R(A) + N(A) \).
Therefore \( R^n = R(A) + N(A) \).

**Proposition 3.6:**
If \( A \) is an idempotent matrix, then \( R^n = R(A) \oplus N(A) \).

**Proof:**
By [Corollary 3.5] we get \( R^n = R(A) + N(A) \), and by [Proposition 2.3] we get \( R(A) \cap N(A) = \{0\} \).
Hence \( R^n = R(A) \oplus N(A) \).

4. Basic properties of non-singular matrices.

In this section we give some basic properties of a non-singular matrices.

**Remark:**
The only nonsingular idempotent matrix is identity matrix \( (I_n) \).
Every idempotent matrix (except \( I_n \)) is singular but a singular matrix may not be idempotent.

**Theorem 4.1 [1]:**
An n×n matrix \( A \) over a number field \( F \) has rank \( n \) if and only if \( A^{-1} \) exists, that is, if and only if \( A \) is non-singular.

**Theorem 4.2 [2]:**
If \( A \) and \( B \) be are idempotent matrices, then the following conditions are equivalent:
1) \( A-B \) is non-singular.
2) \( A+B \) and \( I-AB \) are non-singular.
**Proposition 4.3:**
Let $A$ and $B$ be are idempotent matrices and $AB$ is non-singular, then $N(A) \cap R(B) = \{0\}$.

**Proof:**
Let $\bar{x} \in N(A) \cap R(B)$, then $A \bar{x} = 0$ and $\bar{x} = B \bar{x}$.
So $A \bar{x} = AB \bar{x}$. Hence $AB \bar{x} = 0$.
Therefore $\bar{x} \in N(AB) = \{0\}$ (since $AB$ is non-singular), so $\bar{x} = 0$.
Hence $N(A) \cap R(B) = \{0\}$.

**Theorem 4.4:**
If $A$ and $B$ be are zero commut idempotent matrices, and $A-B$ is non-singular, then $R^n = R(A) \oplus R(B)$.

**Proof:**
Let $\bar{x} \in R(A) \cap R(B)$, then $\bar{x} \in R(A)$ and $\bar{x} \in R(B)$.
Therefore $\bar{x} = A \bar{x}$ and $\bar{x} = B \bar{x}$, so $A \bar{x} = B \bar{x}$, then $(A-B) \bar{x} = 0$.
Hence $\bar{x} \in N(A-B) = \{0\}$ (since $A-B$ is non-singular).
So $\bar{x} = 0$, hence $R(A) \cap R(B) = \{0\}$.
Since $(A-B)(I-A-B) \bar{x} = 0$, then $(I-A-B) \bar{x} \in N(A-B) = \{0\}$, and $(I-A-B) \bar{x} = 0$.
Hence $\bar{x} = A \bar{x} + B \bar{x} \in R(A) + R(B)$ by [Proposition 3.4].
So $R^n = R(A) + R(B)$.
Hence $R^n = R(A) \oplus R(B)$.

**Proposition 4.5:**
If $A$ and $B$ be are zero commut idempotent matrices, then the following conditions are equivalent:
1) $A-B$ is non-singular.
2) $N(A) \cap N(B) = \{0\}$.

**Proof:**
(1$\rightarrow$2) let $\bar{x} \in N(A) \cap N(B)$, then $A \bar{x} = 0$ and $B \bar{x} = 0$.
So $(A-B) \bar{x} = 0$,
therefore $\bar{x} \in N(A-B) = \{0\}$ (since $A-B$ is non-singular).
So $\bar{x} = 0$.
Hence $N(A) \cap N(B) = \{0\}$.
(2$\rightarrow$1) from [Proposition 3.1] we get $N(A) \cap N(B) = N(A-B)$,
but $N(A) \cap N(B) = \{0\}$, then $N(A-B) = \{0\}$.
Hence $A-B$ is non-singular.

**References:**