On $P$-Injective Modules And $\gamma$-Regular Rings

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ABSTRACT

The main purpose of this paper is to study $\gamma$-regular rings, and the connection between such rings and weakly regular, $\pi$-regular, strongly $\pi$-regular, semi $\pi$-regular and SNI-rings. We also study $P$-injective modules, and to find it's relation with $\gamma$-regular rings.

1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unitary. An ideal $I$ of a ring $R$ is called reduced if it contains no non-zero nilpotent elements. For any non empty subset $x$ of a ring $R$, the right (left) annihilator of $x$ will be denoted by $r(x)$ ($l(x)$), respectively. Recall that:

1. A ring $R$ is said to be Von Neumann regular if for every $a \in R$, there exists $b \in R$ such that $a=aba$. The concept of regular rings was introduced by J. Von Neumann in 1936.

2. A ring $R$ is said to be right (left) weakly regular if $a \in aRaR$ ($a \in RaRa$) for every $a \in R$, $R$ is weakly regular ring if it is both right and left weakly regular.

3. A ring $R$ is said to be $\pi$-regular if for every element $a$ in $R$ there exists a positive integer $n=n(a)$ depending on $a$, such that $a^n \in a^nRa^n$. [8]
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(4) A ring $R$ is said to be right semi $\pi$-regular if for each $a$ in $R$, there exist a positive integer $n$ and an element $b$ in $R$ such that $a^n = a^nb$ and $r(a^n) = r(b)$. [7]

(5) A ring $R$ is called a right (left) self-injective (and is denoted by SNI-rings), if and only if, for any essential right (left) ideal $E$ of $R$, every right (left) $R$-homomorphism of $E$ in to $R$ extends to one of $R$ in to $R$. [3]

(6) A ring $R$ is said to be strongly regular if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$.

(7) A ring $R$ is said to be strongly $\pi$-regular if for every $a \in R$, there exists $b \in R$ and a positive integer $n$ such that $a^n = a^{n+1}b$.

(8) Let $I$ be an ideal of a ring $R$. We say that $I$ is pure if for all $x \in I$ there exists $y \in I$ such that $x = xy$. A ring $R$ is called a right (left) semi-duo, if and only if, every principal right (left) ideal of $R$ is a two sided ideal generated by the same element.

(9) A right $R$-module $M$ is said to be $P$-injective, if and only if, for each principal right ideal $I$ of $R$ and every right $R$-homomorphism $f : I \to M$, there exists $y$ in $M$ such that $f(x) = yx$ for all $x \in I$.

(10) A right $R$-module is called GP-injective if for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$, and any right $R$-homomorphism of $a^n R$ in to $M$ extends to one of $R$ in to $M$.

2. $\gamma$-Regular Rings

In this section we introduce the definition of $\gamma$-regular rings and we discuss the connection between $\gamma$-regular rings and other rings which reduced.

Definition 2.1: [5]

An element $a$ of a ring $R$ is said to be $\gamma$-regular if there exists $b$ in $R$ and a positive integer $n \neq 1$ such that $a = ab^n a$. A ring $R$ is said to be $\gamma$-regular if every element of $R$ is $\gamma$-regular element.

Examples 2.2:

The following rings are $\gamma$-regular rings:

1- $Z_3$, $Z_5$, $Z_{11}$, $Z_{15}$

2- Let $R(Z_2)$ be the ring of all 2 by 2 matrices over the ring $Z_2$ (the ring of integer module 2) which are strictly upper triangular.

Clearly, $R(Z_2)$ is $\gamma$-regular ring.

We define a condition (*) as follows:

Definition 2.3: [5]

A ring $R$ satisfies condition (*) if for every $l \neq 1 \in R$ and $b \in R$, there exists a positive integer $m \gg 1$ such that $ab = b^m a$. 

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Theorem 2.4:
Let $R$ be a reduced ring satisfies condition (*) then the following are equivalent:
1- $R$ is a $\mathcal{V}$-regular ring.
2- $R$ is a strongly $\pi$-regular ring.
3- $R$ is $\pi$-regular ring.
Proof:
1$\Rightarrow$ 2: Since $R$ is $\mathcal{V}$-regular ring and satisfies condition (*), then by [5; Theorem 4.4] $R$ is strongly regular ring. So for every $a \in R$ there exists $b \in R$ such that $a=a^2b$. So $(1-ab) \in r(a)$. By [1; Lemma(2.1.9)] $r(a)=r(a^n)$, whence $a^n(1-ab)=0$, then $a^n=a^{n+1}b$. Therefore $R$ is strongly $\pi$-regular ring.

2$\Rightarrow$ 1: For every $a \in R$ there exists $n \in Z^+$ and element $b \in R$ such that $a^n=a^{n+1}b$. Now since $R$ satisfies condition (*), then for every $a,b \in R$, $ab=b^ma$ for some positive integer $m \geq 1$. Then $a^n=a^{n}b^ma$. So $(1-b^ma) \in r(a^n)$. By [1; Lemma(2.1.9)], $r(a^n)=r(a)$, whence $a(1-b^ma)=0$, then $a=ab^ma$. Therefore $R$ is $\mathcal{V}$-regular ring.

1$\Rightarrow$ 3: Since $R$ is $\mathcal{V}$-regular ring and satisfies condition (*), then $R$ is strongly $\pi$-regular ring, and since $R$ is reduced, then by [1; corollary(2.2.7)] $R$ is $\pi$-regular ring.

3$\Rightarrow$ 1: Trivial.

Theorem 2.5:
Let $R$ be a strongly $\pi$-regular ring satisfies condition (*). Then any reduced ideal of $R$ is $\mathcal{V}$-regular.
Proof:
Let $I$ be any reduced ideal of $R$, and let $a \in I$. Since $R$ is strongly $\pi$-regular, there exists a positive integer $n$ and an element $b$ in $R$ such that $a^n=a^{n+1}b$ which implies $a^n(1-ab)=0$ and hence $(1-ab) \in r(a^n)=r(a)=r(a)$, gives $(1-ab)a=0$. Therefore $a=aba$. Now let $c=bab \in I$, then $aca=a.bab.a=aba=a$. Thus $a=aca$, $c \in I$. Consider $(a-a^2c)^2=a^2-a^3c+a^2ca+a^2ca^2c$

$=a^2-a^3c-a(aca)+a(aca)ac$

$=a^2-a^3c-a^2+a^3c
=0$ But $I$ is reduced, then $a-a^2c=0$ implies that $a=a^2c$. Thus $I$ is a strongly regular ideal. Since $R$ satisfies condition (*), then by [5; Theorem 4.4] $I$ is $\mathcal{V}$-regular ideal.

Theorem 2.6:
If $R$ is $\pi$-regular ring satisfies condition (*) and all idempotent elements of $R$ are central, then $R$ is $\mathcal{V}$-regular ring.
Proof:
Since $R$ is $\pi$-regular ring and all idempotent elements of $R$ are central, then by [1; Corollary (2.2.10)] $R$ is strongly $\pi$-regular, and since $R$ satisfies condition (*) then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

Theorem 2.7:
Let $R$ be a left semi-duo ring satisfies condition (*). Then $R$ is $\gamma$-regular if for every $a \in R$, there exists $n \in \mathbb{Z}^+$ such that $a^nR$ is a right semi-regular ideal.

Proof:
Since $R$ is a left semi-duo ring and for every $a \in R$, there exists $n \in \mathbb{Z}^+$ such that $a^nR$ is a right semi-regular ideal, then by [1; Theorem (2.3.8)] $R$ is $\pi$-regular ring, and since $R$ satisfies condition (*) then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

Theorem 2.8:
Let $R$ be a duo ring satisfies condition (*). Then $R$ is $\gamma$-regular if for all $a \in R$, there exists a positive integer $n$ such that $a^nR$ is a pure ideal.

Proof:
Since $R$ be a duo ring and for all $a \in R$, there exists a positive integer $n$ such that $a^nR$ is a pure ideal, then by [1; Theorem (2.3.5)] $R$ is $\pi$-regular ring, and since $R$ satisfies condition (*) then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

Theorem 2.9:
Let $R$ be a reduced ring satisfies condition (*). Then $R$ is $\gamma$-regular iff for all $a \in R$ there exists unit element $k \in R$ and some idempotent $e \in R$ such that $a=ke$.

Proof:
Assume that $R$ is $\gamma$-regular. For any $a \in R$ then there exists $b \in R$ and a positive integer $n \geq 1$ such that $a=ab^n$. Hence $e=ab^n$. Now, we shall prove that $e$ is idempotent element, so $e^2=(ab^n)^2=ab^na=ab^n=e$; and since $R$ is a reduced then $e$ is a central so $a=ab^na=ea=ae$. If we set, $a=e-e+a=a-e^2+ae=(1-e+a)e=ke$. Where $k=(1-e+a)$, then $(1-e+a)(1-e+eb^n)=1-e+eb^n-e^2-e^2b^n+a-ae+aeb^n=1$. So $k=(1-e+a)$ is a unit element in $R$. Therefore $ke=(1-e+a)e=(1-e+a)ab^n=ab^n-eab^n+aab^n=ab^n-ab^n+ae=a$.
Conversely, let $a \in R$, and $a=ke$ for some unit $k \in R$ and some idempotent $e \in R$. Hence $e=ax$ where $x$ the inverse of $k$. Now $ae=aax=keax=kee=ke=a$. Therefore $a=ae=aax$. Since $R$ satisfies condition (*), then for every $a, x \in R$, $ax=x^ma$ for some positive integer $m \geq 1$. Then $a=ax^ma$. Therefore $R$ is $\gamma$-regular ring.
Theorem 2.10:
Let \( R \) be a ring satisfies condition (*) if \( R \) is \( \mathcal{V} \)-regular ring. Then \( R \) is semi-\( \pi \)-regular ring.

Proof:
Let \( R \) be a \( \mathcal{V} \)-regular ring satisfies condition (*). Then by [Theorem 2.4] \( R \) is \( \pi \)-regular ring, then by [1; Theorem (1.3.1)] \( R \) is a right and left semi-\( \pi \)-regular ring.

Example 2.11:
Let \( R(Z_2) \) be the ring of all 2 by 2 matrices over the ring \( Z_2 \) (the ring of integer modulo 2) which are strictly upper triangular. The elements of \( R(Z_2) \) are:
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]
\( R(Z_2) \) is \( \mathcal{V} \)-regular ring satisfies condition (*) clearly, \( R(Z_2) \) is a right semi-\( \pi \)-regular ring, however \( D = D = F \) and \( r(D) = r(C) = r(F) = \{0, A, D, E\} \).

Theorem 2.12:
Let \( R \) be a reduced semi-\( \pi \)-regular ring satisfies condition (*) with every non-zero divisor has inverse. Then \( R \) is \( \mathcal{V} \)-regular ring.

Proof:
Since \( R \) is semi-\( \pi \)-regular ring, then \( r(a^n) = r(e) \) where \( e \) is central idempotent element. Let \( x \in a^n R \cap eR \) implies that \( x = a^n r \), and \( x = er \) for some \( r, r \in R \). Now, see that \( x = er ' = e. er = ex \). Since \( e \in eR = r(a^n) \) then \( a^n e = ea^n = 0 \). Also \( ex = ed^r = 0, \ x = a^n r, \ then \ x = ex = 0 \). Thus \( a^n R \cap eR = 0 \). Now we must prove that \( (a^n + e) \) is non-zero divisor. Let \( (a^n + e)y = 0 \) implies that \( a^n y = -ey \). That is \( a^n y = ey \) \( \in a^n R \cap eR \). Since \( a^n R \cap eR = 0 \). Then \( a^n y = ey = 0 \) and we have \( a^n y = 0 \). That is \( y \in r(a^n) = eR \). There exists \( r_1 \in R \) such that \( y = er_1 \), also \( 0 = ey = e. er_1 = e^2 r_1 = er_1 = y \) (\( e \) is idempotent), since \( (a^n + e) \) is a non-zero divisor. Let \( x \) be the inverse of \( (a^n + e) \). Then we have \( (a^n + e)x = 1 \) implies that \( a^n(a^n + e)x = a^n \) implies \( (a^{2n} + a^n e)x = a^n \). Since \( a^n e = 0 \), then \( a^{2n} x = a^n \) implies \( a^n = a^n a^n x = a^n x a^n \). Since \( R \) satisfies condition (*) then by [Theorem 2.4] \( R \) is \( \mathcal{V} \)-regular ring.

Theorem 2.13:
Let \( R \) be a commutative ring satisfies condition (*), then the following are equivalent:
1- \( R \) is weakly regular ring,
2- \( R \) is \( \mathcal{V} \)-regular ring.

Proof:
1\( \Rightarrow \) 2: Since \( R \) is a commutative weakly regular ring, then by [2; Theorem (1.3.9)] \( R \) is regular ring, and since \( R \) satisfies condition (*), then by [5; Theorem 4.6] \( R \) is \( \mathcal{V} \)-regular ring.
2⇒ 1: Since R is \( \gamma \)-regular ring satisfies condition (*), then by [5; Theorem 4.6] R is regular ring, and since R is commutative then by [2; Theorem (1.3.9)] R is weakly regular ring.

**Theorem 2.14:**

If R is a right SNI-ring satisfies condition (*), then R is \( \gamma \)-regular ring.

**Proof:**

Since R is a right SNI-ring, then by [3; corollary 3.4; P 149] R is weakly regular ring, and since R satisfies condition (*), then by [Theorem 2.13] R is \( \gamma \)-regular ring.

**Theorem 2.15:**

If R is a right SNI-ring which is left self-injective, satisfies condition (*). Then R is \( \gamma \)-regular ring.

**Proof:**

Let R be a right SNI-ring. Then R is semi-prime ring. Thus for any left ideal \( I \), \( L(I) \cap I = 0 \). Since R is SNI-ring, then R is reduced ring, and for any non-zero element \( a \) in R, \( r(a) = L(a) \). Thus \( L(r(a)) \cap L(a) = L(L(a)) \cap L(a) = 0 \), and since R is left self-injective ring, then \( aR \) is a right annihilator and \( R = r(L(r(a))) + r(L(a)) \)\
\( = r(a) + aR \) In particular \( 1 = d + ab \), for some \( b \) in R, and \( d \) in \( r(a) \). Hence \( a = aba \). Since R satisfies condition (*), then by [5; Theorem 4.6] R is \( \gamma \)-regular ring.

**Theorem 2.16:**

Let R be a ring satisfies condition (*), then R is \( \gamma \)-regular if and only if R is a right semi-regular ring.

**Proof:**

Since R is a right semi-regular ring, then by [6; Theorem 3.2] \( r(a) \) is direct summand for every \( a \) in R, and since R satisfies condition (*), then by [5; Proposition 4.11] R is \( \gamma \)-regular ring.

Conversely: If R is \( \gamma \)-regular, then \( r(a) \) is direct summand for every \( a \) in R [3; Proposition 2.17]. Therefore R is a right semi-regular ring [6; Theorem 3.2].

**3. P-Injectivity:**

In this section we discuss the connection between P-injective with \( \gamma \)-regular ring.
Theorem 3.1:
Let $R$ be a ring satisfies condition (*) with every principal right ideal $a^nR$ is a right annihilator generated by the same element, and if $R/a^nR$ is $P$-injective. Then $R$ is $\gamma$-regular ring.

Proof:
Let $a$ be a non-zero element in $R$. Define a right $R$-homomorphism $f:a^nR \rightarrow R/a^nR$ by $f(ax) = x+a^nR$; Clearly $f$ is well defined. Since $R/a^nR$ is $P$-injective, then there exists $c \in R$ such that $f(a^n x) = (c+a^nR)x = ca^n x + a^nR$, for each element $x$ in $R$. In particular; $f(a^n) = 1+a^nR = ca^n + a^nR$, which implies $1-ca^n = a^nR = r(a^n)$. Thus $1-ca^n = r(a^n)$. Therefore, $a^n = ca^n$.

Since $R$ satisfies condition (*) then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

Theorem 3.2:
Let $a$ be an element of a left semi-duo ring, satisfies condition (*), and if $R/a^nR$ is $P$-injective and $a^nR$ is projective. Then $R$ is $\gamma$-regular ring.

Proof:
Let $a \in R$. Define a right $R$-homomorphism $f: R/a^nR \rightarrow a^nR/a^{n+1}R$ by $f(y+a^nR) = a^n y + a^{n+1}R$ for all $y \in R$. Since $a^nR$ is projective, there exists $R$-homomorphism $g: a^nR \rightarrow R/a^nR$ such that $f(g(a^n x)) = a^n x + a^{n+1}R$ for all $x \in R$. But $R/a^nR$ is $P$-injective, then there exists $c \in R$ such that $g(a^n x) = (c+a^nR)a^n x$ then $a^n x + a^{n+1}R = f(g(a^n x)) = f((c+a^nR)a^n x)$ $= f(ca^n x + a^nR)$, (since $a^n x \in a^nR$) $= a^n ca^n x + a^nR$. Since $R$ is a left semi-duo ring, then $ca^n \in R a^n = a^n R$, then $ca^n = a^n r$ for some $r \in R$. So $a^n x + a^{n+1}R = a^n r x + a^{n+1}R$ implies $a^nR = a^{2n}R$. Thus $a^n = a^{2n}d$ for some $d \in R$. Therefore $R$ is strongly $\pi$-regular ring.

Since $R$ satisfies condition (*) then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

Theorem 3.3:
Let $R$ be a reduced ring satisfies condition (*). Then $R$ is $\gamma$-regular ring if $aR/(aR)^2$ is $P$-injective for all $a \in R$ such that $r(a) \subseteq (aR)^2$.

Proof:
Let $aR/(aR)^2$ is $P$-injective ring. Defined $f:aR \rightarrow aR/(aR)^2$ as a right $R$-homomorphism by $f(ax) = ax + (aR)^2$, for all $x$ in $R$, then $f$ is a well define right $R$-homomorphism. Indeed let $x_1, x_2 \in R$ with $ax_1 = ax_2$, implies $(x_1-x_2) \in r(a) \subseteq (aR)^2$, thus $ax_1 + (aR)^2 = ax_2 + (aR)^2$. Hence $f(ax_1) = ax_1 + (aR)^2 = ax_2 + (aR)^2 = f(ax_2)$. Since $aR/(aR)^2$ is $P$-injective, then there exists $c$ in $R$ such that $f(ax) = (ac + (aR)^2)ax = acax + (aR)^2$, for all $x \in R$ yields $a+(aR)^2 = f(a) = aca + (aR)^2$, so $(a-aca) \subseteq (aR)^2$. Since $aca \subseteq (aR)^2$, then $a \subseteq (aR)^2$. Thus $a \subseteq aRaR$. Let $ab_1, ab_2 \in aR$ for any two element $b_1, b_2 \in R$, then $a = ab_1ab_2 = a(b_1b_2) = a(b,b_2) = aca$ for some $c$ in $R$. Thus $a = aca$. Since $R$ satisfies condition (*), then by [5 ; Theorem 4.6] $R$ is $\gamma$-regular ring.
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**Theorem 3.4:**
Let $R$ be a reduced ring satisfies condition (*) with every maximal right ideal is $GP$-injective. Then $R$ is $\gamma$-regular ring.

**Proof:**
Let $a \in R$. We claim first $a^nR+r(a^n)=R$. If not, there exists a maximal right ideal $M$ containing $a^nR+r(a^n)$. Define the canonical injective $f:a^nR \to M$ by $f(a^nb)=a^nb$ for any $b \in R$. Since $M$ is $GP$-injective, then there exists $c \in R$ such that $f(a^nb)=ca^nb$. Therefore $a^n=f(a^n)=ca^n$. Thus $1-c \in L(a^n)=r(a^n) \subseteq M$, which implies $1 \in M$, a contradiction. Hence $a^nR+r(a^n)=R$. In particular $a^nR+r(a^n)$. So $a^nca^n=a^n$. Since $R$ satisfies condition (*), then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

**Theorem 3.5:**
Let $R$ be a duo ring satisfies condition (*). Then $R$ is $\gamma$-regular if for all $a \in R$ there exists a positive integer $n$ such that the principal ideal $a^nR$ is idempotent.

**Proof:**
Let $I$ be an ideal of $R$ such that $I=a^nR$ with $a \in R$ and $n \in Z^+$, assume that $I^n=I$, and let $a^n \in I^n=(a^nR)^n$. Since $R$ is a duo ring, then $a^n \in a^{2n}R$. Hence $a^n=a^{2n}c$ for some $c \in R$. Thus $a^n=a^n.a^nc=a^n.da^n$ for some $d \in R$. Therefore, $R$ is $\pi$-regular ring. Since $R$ satisfies condition (*), then by [Theorem 2.4] $R$ is $\gamma$-regular ring.

**Theorem 3.6:**
Let $R$ be a reduced ring satisfies condition (*). Then $R$ is $\gamma$-regular if every principal right ideal is a right annihilator generated by an element in $R$ and $R/aR$ is $P$-injective ring.

**Proof:**
Let $0 \neq a \in R$. Now, define a right $R$-homomorphism $f:aR \to R/aR$ by $f(ax)=x+aR$ for all $x \in R$, then $f$ is well-defined, indeed, let $ax_1=ax_2$ for any two elements $x_1, x_2$ in $R$, then $a(x_1-x_2)=0$. So $(x_1-x_2) \in R$, then $x_1+aR=x_2+aR$, it mean $f(ax_1)=x_1+aR=x_2+aR=f(ax_2)$. Now, since $R/aR$ is $P$-injective then there exists $c \in R$ such that $f(ax)=(c+aR)ax$ for all $x \in R$. Now, $f(a)=1+aR=ca+aR$, implies $1-ca \in aR=r(a)$. So $1-ca \in r(a)$, whence $a(1-ca)=0$, then $a-aca=0$, so $a=aca$. Since $R$ satisfies condition (*), then by [5; Theorem 4.6] $R$ is $\gamma$-regular ring.
REFERENCES


