ON $\gamma$-REGULAR RINGS

ABDUL AALI J. MOHAMMAD & SANHAN M. SALIH

Department of Mathematics
College of Education
University of Mosul

Received 2006/10/3

ABSTRACT

The main goal of this work is to introduce and study a new type of regular rings called $\gamma$-regular rings. That is, a ring $R$ is said to be $\gamma$-regular if for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = ab^n a$.

We will study some basic properties of those rings including the representation of their elements.

Finally, we will study the relation between $\gamma$-regular rings and other rings.
1 : Introduction:

We conclude that all rings are assumed to be associative with identity.

A ring R is said to be Von Neumann regular if for every \( a \in R \), there exists \( b \in R \) such that \( a = aba \). The concept of regular rings was introduced by J. Von Neumann in 1936[13]. As a generalization of this concept McCoy[1] defined \( \pi \)-regular rings, that is, a ring R with every \( a \in R \), there exists \( b \in R \) and a positive integer \( n \) such that \( a^n = a^n b a^n \). In the recent years regularity and \( \pi \)-regularity have been extensively studied by many authors (cf.[1], [10], [11], [12], [15]).

A ring R is said to be strongly regular if for every \( a \in R \), there exists \( b \in R \), \( a = a^2 b \). This concept has been defined some sixty years ago by R.F.Arens and I.Kaplansky [2], and was studied in recent years by many authors (cf. [3],[4],[9]). It should be noted that in a strongly regular ring R, \( a = ba^2 \) if and only if \( a = a^3 b \) [5]. Azumaya[3] in 1954 defined strongly \( \pi \)-regular rings, that is, a ring R with every \( a \in R \), there exists \( b \in R \) and a positive integer \( n \) such that \( a^n = a^{n+1} b \).

In 1968 Ehrlich [6] defined unit regular rings, that is, a ring R with every \( a \in R \), there exists a unit \( u \in R \) such that \( a = u a u a \).

: \( \gamma \)-Regular Rings:

In this section we introduce the definition of \( \gamma \)-regular rings and some basic properties of them.

Definition 2.1:

An element \( a \) of a ring R is said to be \( \gamma \)-regular if there exists \( b \) in R and a positive integer \( n \not= 1 \) such that \( a = a b^n a \).

A ring R is said to be \( \gamma \)-regular if every element of R is \( \gamma \)-regular element.
ON $\gamma$-REGULAR RINGS

**Examples 2.2:**

The following rings are $\gamma$-regular rings:

1. $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{15}$.
2. $R_{2x2}(\mathbb{Z}_2)$, the ring of 2x2 matrices over $\mathbb{Z}_2$.

It's clear that every $\gamma$-regular ring is regular ring, however the converse is not true in general, for example the rings $(\mathbb{Q}, +, \cdot)$ of rational numbers, the rational(real) Hamilton Quaternion and a quadratic field are regulars but not $\gamma$-regulars because 2 is regular element in each of them but not $\gamma$-regular element.

**Theorem 2.3:**

Let $R$ be a $\gamma$-regular ring and $I$ be an ideal of $R$, then $R/I$ is also $\gamma$-regular ring.

**Proof:** Let $a+I \in R/I$, so $a \in R$. Since $R$ is $\gamma$-regular ring then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a b^n a$.

Hence $a+I = a b^n a+I = (a+I)(b^n+I)(a+I) = (a+I)(b+I)^n(a+I)$.

Therefore $R/I$ is $\gamma$-regular ring.

**Definition 2.4:**

An ideal $I$ of a ring $R$ is said to be $\gamma$-regular if for every element $a \in I$ there exists $b \in I$ and a positive integer $n \neq 1$ such that $a = a b^n a$.

**Definition 2.5:** [7]

A ring $R$ is said to be reduced, if $R$ contains no non-zero nilpotent elements.

**Lemma 2.6:** [7]:

Every idempotent element in a reduced ring is central.

**Proposition 2.7:**

In a reduced $\gamma$-regular ring, every ideal is $\gamma$-regular.
Proof: Let I be any ideal of a reduced \( \gamma \)-regular ring \( R \), and \( a \in I \), then there exists \( b \in R \) and a positive integer \( n \neq 1 \) such that \( a = ab^n a \). Let \( e = ab^n \), then \( e \) is idempotent element and hence it is central.

Now let \( y = ab^{n+1} \), so \( y \in I \), then
\[
ay^n a = a(ay^{n+1}) a = a(ab^{n+1} ab^{n+1} \ldots ab^{n+1}) a = a(ab^n ab^{n+1} ab^{n+1} \ldots ab^{n+1}) a
\]
\[
= a(b ab^n ab^{n+1} \ldots ab^{n+1}) a (Because \ ab^n \ is \ central)
\]
\[
= a(b ab^{n+1} \ldots ab^{n+1}) a = a(b ab^n ab^{n+1} \ldots ab^{n+1}) a
\]
\[
= a(b^2 ab^n ab^{n+1} \ldots ab^{n+1}) a = \ldots = \ldots
\]
\[
= a(b^{n-1} ab^{n+1}) a = a(b^{n-1} ab^n) a = a(b^2 ab^n) a = \ldots = \ldots
\]
\[
= a(ab^n a) a = ab^n a = ab^n a = a. \quad \text{That is} \ a = ay^n a \ \text{and} \ \text{a positive integer} \ n \neq 1 \ \text{Hence} \ I \ \text{is} \ \gamma \text{-regular ideal.}
\]

Theorem 2.8:

A homomorphic image of \( \gamma \)-regular ring is \( \gamma \)-regular ring.

Proof: Let \( f : R \to R' \) be a homomorphism from \( R \) to \( R' \). Let \( y \in f(R) \). Then there exists \( x \in R \) such that \( y = f(x) \). Since \( R \) is \( \gamma \)-regular ring, then there exists \( b \in R \) and a positive integer \( n \neq 1 \) such that \( x = xb^n x \).

Now \( y = f(x) = f(xb^n x) = f(x)f(b^n) f(x) = f(x)(f(b))^n f(x) = y(f(b))^n y \). Therefore \( R \) \( \gamma \)-regular ring.

Lemma 2.9:[10]

If \( R \) is a reduced ring, and if \( a \) is a non-zero element in \( R \). Then \( r(a) = r(a^2) \), and \( l(a) = r(a) \), where \( l(a) \) and \( r(a) \) are the left and right annihilators of \( a \) respectively.

Theorem 2.10:

Let \( R \) be a reduced ring. If \( R/r(a) \) is \( \gamma \)-regular ring for all \( a \in R \), then \( R \) is \( \gamma \)-regular ring.

Proof: Suppose that \( R/r(a) \) is \( \gamma \)-regular ring, then for any \( a + r(a) \in R/r(a) \), there exists \( b + r(a) \in R/r(a) \) and a positive integer \( n \neq 1 \) such that \( a + r(a) = (a + r(a))(b + r(a))^n (a + r(a)) = ab^n a + r(a) \). Then \( a - ab^n a \in r(a) \). So \( a(a - ab^n a) = 0 \).
ON γ-REGULAR RINGS

That is $a^2 (1 - b^na) = 0$. Then $(1 - b^na) \in r(a^2) = r(a)$ [Lemma 2.9].

Lemma 2.11:

If $y$ is an element of a ring $R$ such that $a - ay^a$ is $γ$-regular element, then $a$ is a regular element, where $1 \neq x$ is a positive integer.

Proof: Suppose that $a - ay^a$ is $γ$-regular element, then there exists an element $b \in R$ and a positive integer $n \neq 1$ such that

$$a - ay^a = (a - ay^a) b^n (a - ay^a).$$

Now $a - ay^a = (a - ay^a) (b^n a - b^n ay^a) = ab^na - ab^nay^a - ay^ab^na + ay^ab^nay^a$,

then $a = ay^a + ab^na - ab^nay^a - ay^ab^na + ay^ab^nay^a$

$$= a(y^a + b^n - b^ny^a - y^ab^n + y^ab^nay^a)a = az_a.$$

Where $z = y^a + b^n - b^ny^a - y^ab^n + y^ab^nay^a$. Therefore $a$ is a regular element.

Theorem 2.12:

Let $R$ be a ring and let $I$ be a $γ$-regular ideal such that $R/I$ is $γ$-regular. Then $R$ is regular ring.

Proof: Let $a \in R$. Then $a + I \in R/I$. Since $R/I$ is $γ$-regular ring, then there exists $b + I \in R/I$ and a positive integer $n \neq 1$ such that $a + I = (a + I)(b + I)^n(a + I)$. Then $a + I = ab^na + I$. So $a - ab^na \in I$. Hence $a$ is regular element [Lemma 2.11]. Therefore $R$ is regular ring.

Definition 2.13:

A ring $R$ is said to be unit $γ$-regular if for every $a$ in $R$ there exists a unit $u$ in $R$ and a positive integer $n \neq 1$ such that $a = a u^n a$.

Definition 2.14:[12]

A ring $R$ is said to be a semi-commutative ring if every idempotent element in $R$ is central.

Hence every reduced ring is semi-commutative ring.[12]
Theorem 2.15:

Let \( R \) be a semi-commutative \( \gamma \)-regular ring, then \( R \) is unit regular ring.

**Proof:** Let \( x \in R \), then there exists \( y \in R \) and a positive integer \( n \neq 1 \) such that \( x = xy^n x \). Then \( xy^n \) and \( y^n x \) are idempotent elements.

Hence \( xy^n = x(y^n x)y^n = (xy^n)(y^n x) = y^n(xy^n x) = y^n x \). Let \( v = xy^n + xy^{2n}-1 \) and \( u = x + xy^n - 1 \). Since \( xy^n = y^n x \) and \( x = xy^n x \), we have
\[
uv = (x + xy^n - 1)(xy^n + xy^{2n}-1)
= xxy^n + xxy^{2n} - x + xxy^nxy^n + xxy^nxy^{2n} - xy^n - xy^n - xy^{2n} + 1
= xy^n x + xy^nxy^n - x + xxy^nxy^n + xxy^nxy^n - xy^n - xy^n - xy^n + 1
= x + xy^n - x + xy^n - xy^n - xy^n = 1.
\]

Similarly \( vu = 1 \) and \( xvv = x(xy^n + xy^{2n} - 1)x = xxy^n x + xxy^n y^n x - x^2 = x^2 + x - x^2 = x \).

Therefore \( R \) is unit regular ring.

Proposition 2.16:

If \( R \) is a ring such that for each nonzero element \( a \in R \) there is a unique \( b \in R \) such that \( a^n = a^n ba^n \), a positive integer \( n \neq 1 \), then \( b \) is \( \gamma \)-regular element.

**Proof:** Since \( a^n = a^n ba^n \) for each \( a \in R \), then \( R \) has no divisor of zero. Then cancellation law holds.

\( ow \, 1 = ba^n \Rightarrow b = ba^nb \). Therefore \( b \) is \( \gamma \)-regular element.

Proposition 2.17:

If a ring \( R \) is \( \gamma \)-regular, then \( r(a) \) is direct summand for every \( a \) in \( R \).

**Proof:** Since \( R \) is \( \gamma \)-regular, then for each \( a \in R \) there exists \( b \in R \) and a positive integer \( n \neq 1 \) such that \( a = ab^na \). Then \( a(1 - b^na) = 0 \).

So \( 1 - b^na \in r(a) \). Hence \( 1 - d \in r(a) \), where \( d = b^na \) and \( ad = a \).

Now \( 1 = d + (1 - d) \), then \( R = dR + r(a) \). We shall prove that \( dR \cap r(a) = 0 \).

Let \( x \in dR \cap r(a) \), then \( x \in dR \) and \( ax = 0 \). So \( x = dc \) for some \( c \in R \).

Now \( ax = adc = 0 \), then \( ac = 0 \). So \( b^ac = 0 \). That is \( dc = 0 \). Hence \( x = 0 \).
ON $\gamma$-REGULAR RINGS

Now $y = u^1 ay = a(u^1 y) \in aR$ (Because $a = e + u$ and $R$ is semi commutative, $e^1 a = u^1 e + 1$ and $a u^1 e = e u^1 + 1 = u^1 e + 1$. So $a u^1 = u^1 a$). Hence $(1-e)R \subseteq aR$.

4: $\gamma$-Regular Rings with condition (*):

One of the most important rings was introduced by Kandasamy [14] is quasi-commutative rings that is a ring $R$ with $ab = b^m a$ for every pair $a, b \in R$ and for some positive integer $m$.

Here we restrict the quasi-commutative ring to the condition that has a main role in our proofs and to discuss the connection between $\gamma$-regular rings and some other rings. The condition is (*): Let $R$ be a ring such that for every $1 \neq a \in R$ and $b \in R$, there exists a positive integer $m > 1$ such that $ab = b^m a$.

The reason for $1$ that not satisfies condition (*) is $1 \cdot b = b^m \cdot 1$, this equation is true if $m = 1$, also the identity element $1 \in R$ is $\gamma$-regular element, strongly $\gamma$-regular element.

In this section we discuss the connection between $\gamma$-regular ring with the other rings which they are commutative, reduced or satisfies condition (*).

**Proposition 4.1:**

Every reduced $\gamma$-regular ring is strongly regular ring.

**Proof:** Since reduced $\gamma$-regular ring implies reduced regular ring, then it’s strongly regular ring, [10; Theorem 1.3.7].

But the converse of this theorem is not true in general. For example the ring $(Q, +, \cdot)$ of rational number is reduced strongly regular but not $\gamma$-regular.

**Corollary 4.2:**
Let $R$ be a semi-commutative $\gamma$-regular ring. Then $R$ is strongly regular ring.

**Proof:** Follows from [12; Proposition 1.2.5].

**Corollary 4.3:**

If $R$ is duo $\gamma$-regular ring, then $R$ is strongly regular ring.

**Theorem 4.4:**

Let $R$ be a ring satisfies condition (*), then the following are equivalent:

1- $R$ is $\gamma$-regular ring.

2- $R$ is strongly regular ring.

**Proof:** $1 \Rightarrow 2$: For every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = ab^na$. Since $R$ satisfies condition (*), then $ab^n = (b^n)^{ra} = b^{mr}a$ for some positive integer $r > 1$.

Now $a = ab^n = b^{mr}a$, $a = b^{nr}a$, $c = b^{nr} \in R$, and if $b^na = a^mb^n$ for some positive integer $m > 1$, then $a = aa^{mb^n} = a^2 a^{m-1}b^n = a^2 d$, where $d = a^{m-1}b^n \in R$.

Therefore $R$ is strongly regular ring.

$2 \Rightarrow 1$: For every $a \in R$ there exists $b \in R$ such that $a = a^2 b$. Now $a = a.ab = ab^n a$ for some positive integer $n > 1$ ($R$ satisfies condition (*)). Then $R$ is regular ring.

From the proof of above theorem also we can shows that $1 \Rightarrow 2$ even when $R$ is quasi-commutative ring, as in the following corollary:

**Corollary 4.5:**

Let $R$ be a quasi-commutative $\gamma$-regular ring, then $R$ is strongly regular ring.

**Theorem 4.6:**
ON $\gamma$-REGULAR RINGS

Let $R$ be a reduced ring satisfies condition (*). Then the following are equivalent:

1- $R$ is $\gamma$-regular ring.
2- $R$ is regular ring.

**Proof**: $1 \Rightarrow 2$: Clearly from the definition of $\gamma$-regular ring.

$2 \Rightarrow 1$: Since $R$ is reduced and regular then $R$ is strongly regular ring, and $y$ [Theorem 4.4] $R$ is $\gamma$-regular ring.

**Theorem 4.7**: 

Let $R$ be a ring satisfies condition (*). Then $R$ is $\gamma$-regular ring if and only if every principal ideal of $R$ is generated by an idempotent.

**Proof**: If $R$ is $\gamma$-regular ring, then its clearly that every principal right ideal of $R$ is generated by an idempotent $,[13]$.

Conversely: If $aR=eR$, where $e$ is an idempotent element. Then $a=er$ for some $r$ in $R$. Now $a=er=e^2r=ea$. Let $c=ab$ for some $b$ in $R$, since $R$ satisfies condition (*), then $c=b^na$ for some positive integer $n > 1$.

Now $a=ea=b^na.a=b^na^2$. Similarly for $RaRe$. Then $R$ is strongly regular. therefore $R$ is $\gamma$-regular ring $,[Theorem 4.4]$.

**Proposition 4.8**: 

Let $R$ be a ring satisfies condition (*). Then the following are equivalent:

1- $R$ is $\gamma$-regular ring.
2- Every principal ideal is a direct summand.

**Proof**: $1 \Rightarrow 2$: Clearly, from $[4; Proposition 1.1.3]$.

$2 \Rightarrow 1$: Let $R=aR \oplus K$ for some ideal $K$ of $R$, it's clear that $a \neq 1$ because if $a=1$ then $R=1 \cdot R \oplus \{0\}$, the proof being trivial. Since $1 \in R$ then $1=ar+k$ for some $r \in R$ and $k \in K$. Since $R$ satisfies condition (*) then $ar=r^na$; and a positive integer $n \neq 1$. 

---

130
Then \(1 = r^n a + k \) imply \(a = a r^n a + ak \) and \(ak \in aR \cap K = 0\). So \(a = a r^n a\). Therefore \(R\) is \(\gamma\)-regular ring.

**Remark 4.9:**

If we add the condition that \(R\) satisfies condition (*) in [Corollary 3.3], then the converse holds.

**Theorem 4.10:**

If \(R\) is a reduced ring satisfies condition (*). Then \(R\) is \(\gamma\)-regular ring if and only if for every element \(a \in R\), \(a = eu\), where \(e\) is an idempotent and \(u\) is unit.

**Proof:** If \(R\) is reduced \(\gamma\)-regular ring, then \(a = eu\), where \(e\) is an idempotent and \(u\) is unit [Corollary 3.3].

Conversely: Let \(a = ue\), where \(e\) is idempotent and \(u\) is unit, then \(e = ra\) where \(r\) is the inverse of \(u\). Now \(ae = ara\), but \(ae = ue, e = ue^2 = a\), then \(a = ara\) which is regular. Therefore, \(y\) [Theorem 4.6] \(R\) is \(\gamma\)-regular ring.

**Proposition 4.11:**

Let \(R\) be a ring satisfies condition (*), then \(R\) is \(\gamma\)-regular if and only if \(r(a)\) is direct summand for every \(a\) in \(R\).

**Proof:** If \(R\) is \(\gamma\)-regular, then \(r(a)\) is direct summand for every \(a\) in \(R\) [Proposition 2.17].

Conversely: Let \(R = aR + r(a)\). In particular \(1 = ar + d\), where \(r \in R\) and \(d \in r(a)\), then \(a = a^2 r + ad\) imply \(a = a^2 r\) and hence \(a = a^2 r = aar = ar^n a\) for some positive integer \(n > 1\). Therefore \(R\) is \(\gamma\)-regular ring.

**Theorem 4.12:**

Let \(R\) be a ring satisfies condition (*). Then the following are equivalent:

1. \(R\) is \(\gamma\)-regular ring.
ON $\gamma$-REGULAR RINGS

2- For every right ideal $I$ and left ideal $J$ in $R$, $IJ=I\cap J$.

**Proof:** $1 \Rightarrow 2$: Since every $\gamma$-regular is regular, then 2 holds by [10; Theorem 1.1.7].

$2 \Rightarrow 1$: Let $x$ in $R$, since $x$ is in $xR \cap Rx=xRx$, there is an element $y$ in $R$ such that $xyx=x$. Since $R$ satisfies condition (*), then $x=y^nx^2$ for some positive integer $n > 1$. So $R$ is strongly regular. By [Theorem 4.4] $R$ is $\gamma$-regular ring.

**Theorem 4.13:**

Let $R$ be a reduced ring satisfies condition (*). If every ideal of $R$ is a maximal right ideal, then $R$ is $\gamma$-regular ring.

**Proof:** Since $R$ is reduced and every prime ideal is maximal right ideal, then $R$ is regular ring [8], and since $R$ satisfies condition (*) and is reduced, then by [Theorem 4.6] $R$ is $\gamma$-regular ring.

**Corollary 4.14:**

Let $R$ be a reduced ring satisfies condition (*). Then $R$ is a $\gamma$-regular ring if $R/P$ is $\gamma$-regular ring for every prime ideal $P$ in $R$.

**Proof:** Let $P$ be a prime ideal in $R$, then $R/P$ is a division ring, because $R/P$ is a $\gamma$-regular and has no nonzero divisor. Therefore $P$ is maximal right(left)ideal in $R$ and $R$ is a $\gamma$-regular ring[Theorem 4.13].

**Theorem 4.15:**

Let $R$ be a ring satisfies condition (*). Then $R$ is $\gamma$-regular ring if and only if $1 = \sqrt{1}$ for each ideal $I$ in $R$.

**Proof:** Suppose that $R$ is $\gamma$-regular ring, its clearly that $1 \subseteq \sqrt{1}$ for each ideal $I$ in $R$. Now let $b \in \sqrt{1}$ then $b^n \in I$ for some $n \in \mathbb{Z}^+$, then there exists $c \in R$ and $1 \neq r \in \mathbb{Z}^+$ such that $b^n = b^n c^r b^n$. Since $R$ satisfies condition (*), then $b^n c^r = (c^r)^n b^n$ for some positive integer $m > 1$. That is $b^n = c^m b^{2n}$. So $b^{n-1}$ can be
ABDUL AALI J. MOHAMMAD & SANHAN M. SALIH

written in the form \( b^{n-1} = c^{mr} b^{2(n-1)} = c^{mr} b^n b^{n-2} \in I \), and we repeat this \( n \)-times we get \( b \in I \), then \( \sqrt{I} \subseteq I \). Hence \( I = \sqrt{I} \).

Conversely: Let \( I = \sqrt{I} \) for each ideal \( I \) in \( R \). Take \( 1 = a^2 R = \sqrt{a^2 R} \) then \( a^2 \in a^2 R \Rightarrow a \in \sqrt{a^2 R} = a^2 R \Rightarrow a \in a^2 R \). Hence \( R \) is strongly regular ring. Therefore by [Theorem 4.4] \( R \) is \( \gamma \)-regular ring.

**Corollary 4.16:**

Let \( R \) be a ring satisfies condition (*) Then \( R \) is \( \gamma \)-regular ring if and only if each ideal \( I \) in \( R \) is semi-prime.

**Theorem 4.17:**

If \( R \) is a reduced ring satisfies condition (*) and every maximal ideal of \( R \) is a right annihilator, then \( R \) is \( \gamma \)-regular ring.

**Proof:** Let \( a \in R \), we shall prove that \( aR + r(a) = R \). If not, there exists a maximal right ideal \( M \) containing \( aR + r(a) \). If \( M = r(b) \) for some \( 0 \neq b \in R \), we have \( b \in l(a) \subseteq l(a) = r(a) \) [10; Theorem 1.3.10], which implies \( b \in M = r(b) \), then \( b^2 = 0 \) and \( b = 0 \), a contradiction. Therefore \( aR + r(a) = R \). In particular, \( ac + d = 1 \), with \( c \in R \) and \( d \in r(a) \), then \( a^2 c + ad = a \) implies \( a^2 c = a \), then \( R \) is strongly regular ring. Therefore \( R \) is \( \gamma \)-regular ring [Theorem 4.4].

**Theorem 4.18:**

Let \( R \) is a reduced ring satisfies condition (*) such that every principal right ideal of \( R \) is a right annihilator, then \( R \) is \( \gamma \)-regular ring.

**Proof:** Since \( R \) is reduced and every principal right ideal of \( R \) is a right annihilator, then by [10; Theorem 1.3.10] \( R \) is strongly regular ring, and since \( R \) satisfies condition (*) then by [Theorem 4.4] \( R \) is \( \gamma \)-regular ring.
ON $\gamma$-REGULAR RINGS

5: Strongly $\gamma$-Regular Rings:

In this section we introduce another new type of rings that [Proposition 4.1], [Corollary 4.2] and [Corollary 4.3] leads us to define it and we shall call those rings as a strongly $\gamma$-regular rings.

Definition 5.1:

Let $R$ be any ring. Then $R$ is called rig strongly $\gamma$-regular ring if for every element $a \in R$, there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2 b^n$.

Hence, in a strongly $\gamma$-regular ring $R$, $a = a^2 b^n$ if and only if $a = b^n a^2$, [5].

In a commutative ring, the equation $ab^n a = a$ may be written as $a^2 b^n = a$.

That is, a commutative ring $R$ is $\gamma$-regular if and only if it is strongly $\gamma$-regular. We see that every strongly $\gamma$-regular ring is strongly regular ring, however the converse is not true in general, for examples the rings $(Q, +, \cdot)$ of rational numbers, the rational(real) Hamilton Quaternion and a quadratic field are strongly regulars but not strongly $\gamma$-regulars.

Theorem 5.2:

Let $R$ be a strongly $\gamma$-regular ring and $I$ be an ideal of $R$. Then $R/I$ is also strongly $\gamma$-regular ring.

Proof: Let $a + I \in R/I$, so $a \in R$. Since $R$ is strongly $\gamma$-regular ring then there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2 b^n$.

Hence $a + I = a^2 b^n + I = (a^2 + I)(b^n + I) = (a + I)^2(b + I)^n$. Therefore $R/I$ is strongly $\gamma$-regular ring.

Theorem 5.3:

A homomorphic image of strongly $\gamma$-regular ring is strongly $\gamma$-regular ring.

Proof: The proof is similar to the proof of Theorem [2.8].
Here we want to find the condition for strongly regular ring to be strongly $\gamma$-regular ring.

**Theorem 5.4:**

Let $R$ be a ring satisfies condition (*), then the following are equivalent:
1. $R$ is strongly $\gamma$-regular ring.
2. $R$ is strongly regular ring.

**Proof:** 1$\Rightarrow$2: Clearly from the definition of strongly $\gamma$-regular ring.
2$\Rightarrow$1: Since $R$ is strongly regular ring, then for every $a \in R$, there exists $b \in R$ such that $a = a^2b$.

Now since $R$ satisfies condition (*), then for every $a, b \in R$, $ab = b^na$ for some positive integer $n > 1$. Then $a = ab^na$, and since $R$ is strongly regular then $R$ is reduced, implies $a = a^2b^n$. Therefore $R$ is strongly $\gamma$-regular ring.

**Theorem 5.5:**

If $R$ is a regular ring satisfies condition (*), then $R$ is strongly $\gamma$-regular ring.

**Proof:** Since $R$ is regular ring, then for every $a \in R$, there exists $b \in R$ such that $a = aba$. Since $R$ satisfies condition (*), then $ab = b^na$ with a positive integer $n \neq 1$ for every $a, b \in R$. Then $a = a^2b^n$. Therefore $R$ is strongly $\gamma$-regular ring.

Here we lead to discuss the connection between $\gamma$-regular rings and strongly $\gamma$-regular rings.

**Theorem 5.6:** Every strongly $\gamma$-regular ring is $\gamma$-regular ring.

**Proof:** Since $R$ is strongly $\gamma$-regular ring, then for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2b^n = b^na^2$.

Now if $a = b^na^2$, then $ab = (b^na^2)b^n = b^n(b^2a^n) = b^na$. This implies $ab = c^na^2 = a$, then $a = ab^na$. Therefore $R$ is $\gamma$-regular ring.
ON $\gamma$-REGULAR RINGS

The converse of this theorem is not true in general. For example the ring $\mathbb{R}_{2\times 2}(\mathbb{Z}_2)$ of $2\times 2$ matrices over the ring $\mathbb{Z}_2$ is $\gamma$-regular ring but not strongly $\gamma$-regular ring because the element $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is $\gamma$-regular element but not strongly $\gamma$-regular element.

Theorem 5.7:

Let $R$ be a ring. Then $R$ is strongly $\gamma$-regular if and only if $R$ is reduced $\gamma$-regular.

Proof: Suppose that $R$ is reduced $\gamma$-regular, then for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = ab^n a$.

Now $(a - a^2 b^n)^2 = (a - a^2 b^n)(a - a^2 b^n) = a^2 - a^3 b^n - a^2 b^n a + a^2 b^n a^2 b^n = a^2 - a^3 b^n - a^2 b^n a + a^2 b^n a + a^2 b^n a^2 b^n = a^2 - a^3 b^n - a + a^2 b^n a^2 b^n = 0$. Since $R$ is reduced then $a - a^2 b^n = 0$, and then $a = a^2 b^n$. Similarly $(a - b^n a^2)^2 = 0$ which implies $a = b^n a^2$. Therefore $R$ is strongly $\gamma$-regular.

Conversely: Suppose that $R$ is strongly $\gamma$-regular ring, then by [Theorem 5.6] $R$ is $\gamma$-regular ring. To prove that $R$ is a reduced ring, suppose that there exists a positive integer $n$, such that $c^n = 0$ for some $c \in R$. Since $c = c^2 d^m$ for sum positive integer $m \neq 1$ gives $0 = c_n d^m = c^{n-1}$ and $0 = c^n d^m = c^{n-2}$ and so on $c = 0$, then $R$ is a reduced ring. Therefore $R$ is a reduced $\gamma$-regular ring.

Corollary 5.8:

If $R$ is a strongly $\gamma$-regular ring, then $R$ is a unit regular ring.

Proof: Since every strongly $\gamma$-regular is reduced $\gamma$-regular [Theorem 5.7], and since every reduced ring is semi-commutative ring, then by [Theorem 2.15] $R$ is a unit regular ring.
Corollary 5.9:

If a ring $R$ is strongly $\gamma$-regular, then $a^n$ is a unit $\gamma$-regular element for each $a \in R$ and positive integer $n > 1$.

Proof: Let $R$ be a strongly $\gamma$-regular, then $R$ is $\gamma$-regular and reduced [Theorem 5.7], and hence $R$ is semi-commutative ring. Therefore by [Theorem 3.1] $a^n$ is a unit $\gamma$-regular element. ■

Theorem 5.10:

Let $R$ be a ring. If $R$ is semi-commutative $\gamma$-regular ring, then $R/N$ is strongly $\gamma$-regular ring.

Proof: since $R$ is $\gamma$-regular then for each $a \in R$, there exists $b \in R$ and a positive integer $m \neq 1$ such that $a = ab^m$. Let $e = ab^m$, then $e$ is idempotent and hence central, then $a = ae = ea$.

Now $a(1-e) = 0$ implies $(a(1-e))^n = 0$, this means that $a(1-e) \in N$, so $a + N = ae + N$.

Thus $a + N = aab^m + N = a^2b^m + N$, yielding $a + N = (a^2 + N)(b^m + N)$. Therefore $R/N$ is strongly $\gamma$-regular ring. ■

Lemma 5.11:

Let $R$ be a strongly $\gamma$-regular ring. Then $R$ is semi-commutative ring.

Proof: From [Theorem 5.7]. ■

Proposition 5.12:

Let $R$ be a semi-commutative $\gamma$-regular ring, then $R$ is strongly $\gamma$-regular ring.

Proof: Let $R$ be a $\gamma$-regular ring, and let $a$ be a non-zero element in $R$, then there exists $b$ in $R$ and a positive integer $n \neq 1$ such that $a = ab^n$.

Let $e = ab^n$, then $e$ is an idempotent element, and hence $e$ is central (Since $R$ is semi-commutative ring). So $a = ea = ae = a^2b^n$, and if $e = b^n a$ then $e$ is also an idempotent element, and hence $e$ is central. So $a = ae = ea = b^na^2$. Therefore $R$ is a strongly $\gamma$-regular ring. ■
**Corollary 5.13:**

Let $R$ be a duo $\gamma$-regular ring, then $R$ is strongly $\gamma$-regular ring.

**Proof:** Since $R$ is duo ring, then every idempotent element is central [18; Lemma 1.1.9]. Hence $R$ is semi-commutative ring. Also $R$ is a $\gamma$-regular ring, then by [Proposition 5.12], $R$ is strongly $\gamma$-regular ring.

From [Theorem 2.10] we conclude the following:

**Theorem 5.14:**

Let $R$ be a reduced ring. If $R/\tau(a)$ is $\gamma$-regular ring for all $a \in R$, then $R$ is strongly $\gamma$-regular ring.

**Proof:** Suppose that $R/\tau(a)$ is $\gamma$-regular ring, then for any $a+\tau(a) \in R/\tau(a)$, there exists $b+\tau(a) \in R/\tau(a)$ and a positive integer $n \neq 1$ such that $a+\tau(a) = (a+\tau(a))(b+\tau(a))^n(a+\tau(a)) = ab^n a + \tau(a)$. Then $a-ab^n a \in \tau(a) \Rightarrow a(a-ab^n a) = 0 \Rightarrow a^2(1-b^n a) = 0$.

Then $(1-b^n a) \in \tau(a^2) = \tau(a) = l(a)$ [4: Lemma 1.3.6 and 4: Lemma 1.3.4] $\Rightarrow (1-b^n a)a = 0 \Rightarrow a = b^n a$, and from $(1-b^n a)a = 0 \Rightarrow (1-b^n a) \in l(a) = \tau(a)$ [Lemma 1.2.13], then $a-ab^n a = 0 \Rightarrow (1-ab^n)a = 0 \Rightarrow (1-ab^n) \in l(a) = \tau(a) \Rightarrow a(1-ab^n) = 0$ then $a = a^n b^n$.

Therefore $R$ is strongly $\gamma$-regular ring.

**Theorem 5.15:**

Let $R$ be a strongly $\pi$-regular ring. Then $a^n$ is strongly $\gamma$-regular elements for every $a \in R$ and $n \in \mathbb{Z}^+$.

**Proof:** Let $R$ be a strongly $\pi$-regular ring, then for every $a \in R$ there exists $b \in R$ and $n \in \mathbb{Z}^+$ such that $a^n = a^{2n} b^n$.

Hence $a^n$ is strongly $\gamma$-regular element.
REFERENCES


