Algorithms of q-Schur Algebra

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الخلاصة

في العام 1989 عرف نوع جديد من الجبر بـ شـور مـن النـوع q ويرمـز لـه بـ $S_q(n,r)$ من قبل ديبر وجيمس في [2] وهو تعميم لجبر شور في حالة q=1 والمقدم مـن قبل كرين في [8]. ان واحدة من أهم المشاكل التي لم تحل لحد الآن هي ايجاد صيغة رياضية مناسبة لحاصل ضرب عنصرين من $S_q(n,r)$ باستثناء الصيغة الرياضية والتي قدمها كـرين في [8] والصيغة البرمجية التي دعمت هذه الصيغة والمقدمة من قبل محمود في [11] عندما q=1.

في هذا البحث سنأخذ q بحالتها العامة، محاولين ايجاد هذه الصيغة الرياضـية غيـر المعروفة لحد الآن. ان حل هذه المشكلة يتطلب فك لغز معضلتين، بالنسبة للمعضلة الاولـى سنقدم حلاً جبرياً وعددياً عندما n=2 و r≥2 وحلاً برمجياً عندما r,n≥2. آملين تماماً من حل المعضلة الثانية والتي تتطلب جهداً ووقتاً أطول من الاولى في وقت آخر.

Abstract

In 1989, Dipper and James [2] extend the notion of a Schur algebra, [see Green 8], to obtain the q-Schur algebra $S_q(n,r)$. One of the main difficulties that can not be solved is to determine a suitable mathematical formula for the product between two elements in $S_q(n,r)$, except for the mathematical formula introduced by Green in [8] and the programming formula that supported this formula which is given by [11] when q=1.

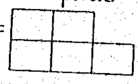
In this paper, we will take q in general, trying to find the mathematical formula which is not known till now. The solution for this problem needs two things, the first one, we introduce an algebraic and mathematical solution when n=2 and r≥2, and the programming solution when n,r≥2. The second one, we hope to find the solution for it later.

1.Introduction

A composition μ for r is a sequence $(\mu_1, \mu_2, \ldots)$ of non-negative integers such that $|\mu| = \sum_i \mu_i = r$. The integers μ_i, for $i \geq 1$, are the parts of μ; if $\mu_i=0$ for $i>m$, we identify μ with $(\mu_1, \mu_2, \ldots, \mu_m)$. A composition μ is a partition if $\mu_i \geq \mu_{i+1}$ for all $i \geq 1$.

The diagram of Young of a composition μ is the subset

$$[\mu] = \{(x, y) \mid 1 \leq y \leq \mu_x \quad \text{and} \quad x \geq 1\} \text{ of } N \times N.$$

The elements of $[\mu]$ are called the nodes of μ; more generally, a node is any element of N×N. It is useful to represent the diagram of μ as an array of boxes in the plane. For example, if $\mu=(2,3)$ then $[\mu]=$ 

If μ is a composition of n then a μ-tableau is a bijection

$$t: [\mu] \rightarrow \{1,2,\ldots,n\}$$

A μ-tableau **t** is row standard (*resp. row semi-standard*) if the entries in **t** increase from left to right in each row (*resp. if the entries in each row in **t** are non-decreasing*), **t** is standard (*resp. semi-standard*) if μ is a partition and the entries in **t** increase from left to right in each row and from top to bottom in each column (*resp. μ is a partition, **t** is row semi-standard and the entries in each column of **t** are strictly increasing*).

More information about the number of tableaux found in [10].

Given any composition μ, let t^μ be the row standard μ-tableau in which the integers 1,2,…,r are entered in increasing order from left to right along the rows of $[\mu]$. For example, if $\mu=(2,3)$, then

$$t^\mu = \begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

If $\mu=(\mu_1, \ldots, \mu_n)$ is a composition of r, then the symmetric group G_r acts from the right upon the set of μ-tableaux by permuting the entries inside a given tableau. For example

$$t = \begin{array}{|c|c|c|}\hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline\end{array} \qquad \text{, then } t(2,3,5) = \begin{array}{|c|c|c|}\hline 3 & 4 & \\ \hline 1 & 5 & 2 \\ \hline\end{array}$$

2.q-Schur algebra

We fix an integer $r \geq 1$ and the symmetric group G_r acting on 1,2,…,r from the right. For i=1,2,…,r-1 , let s_i be the basic transposition (i, i+1) and let $S=\{s_1, s_2, \ldots, s_{r-1}\}$. Then, as a coxeter group [see 7], G_r is generated by $s_1, s_2, \ldots, s_{r-1}$, subject to the relations

$$\left.\begin{array}{lll} s_i^2 = 1 & \text{for} & i = 1,2,\ldots,r-1, \\ s_i s_j = s_j s_i & \text{for} & 1 \leq i < j-1 \leq r-2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for} & i = 1,2,\ldots,r-2 \end{array}\right\} \ldots (2.1)$$

Suppose that w is an element of G_r and write $w = s_{i_1} \ldots s_{i_k}$ where $s_{i_1}, \ldots, s_{i_k}$ are elements of S. If k is minimal we say that w has length k and write l(w)=k. In this case, $s_{i_1}, \ldots, s_{i_k}$ is called a reduce expression for w.

Now, let R be a commutative domain with 1 and let q be an arbitrary element of R. The Iwahori-Hecke algebra $H=H_{R,q}(G_r)$ of G_r; [see 5], is the unital associative R-algebra with generators $T_1, T_2, \ldots, T_{r-1}$ and relations:

$$\left.\begin{array}{lll} (T_i - q)(T_i + 1) = 0 & \text{for} & i = 1,2,\ldots,r-1, \\ T_i T_j = T_j T_i & \text{for} & 1 \leq i < j-1 \leq r-2, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for} & i = 1,2,\ldots,r-2 \end{array}\right\} \ldots (2.2)$$

In particular, when q=1, the first relation in (2.2) reduces to $T_i^2 = 1$. By the reduced expression for w, define $T_w = T_{i_1} \ldots T_{i_k}$.

Dipper and James in [3] proved the following relations:

Lemma (2.3): Suppose that $s \in S$ and $w \in G_r$. Then

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w & \text{if } l(ws) < l(w) \end{cases}$$

Theorem (2.4): The Iwahori-Hecke algebra H is free as an R-module with basis $\{T_w | w \in G_r\}$.

The Young subgroup $G_\mu = G_{\mu_1} \times \ldots \times G_{\mu_n}$ of G_r is the row stabilizer of t^μ. Then G_μ is generated by $S \cap G_\mu$, since $s_i \in G_\mu$ if and only if i and i+1 are in the same row of t^μ. Let $H(G_\mu)$ be the sub-algebra of H

generated by $\{T_s | s \in S \cap G_\mu\}$. By Lemma (2.3) and Theorem (2.4), $H(G_\mu)$ is free as an R-module with basis $\{T_w | w \in G_r\}$. Consequently, $H(G_\mu) \cong H(G_{\mu_1}) \times \ldots \times H(G_{\mu_n})$.

For example, if μ=(2,3), $t^\mu = \begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$, then

$$G_\mu = G_{\mu_1} \times G_{\mu_2} = G_2 G_3 = < s_1, s_3, s_4 > .$$

Mathas in [12], propose the following proposition:

Proposition (2.5): Suppose that μ is a composition of r and let $D_\mu = \{d \in G_r \mid t^\mu d \text{ is row standard}\}$. Then D_μ is a complete set of right coset representatives of G_μ in G_r.

Definition (2.6): Let t be a tableau of type θ and let γ be a composition. Then $\gamma(t)$ is the tableau of type γ obtained from t by replacing each entry i in t by j if i appears in row j of t^γ.

For example, let $t = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$ and γ=(3,2,1). Then

$$t^\gamma = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \text{and} \quad \gamma(t) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} .$$

Definition (2.7): Let μ and γ be compositions.Then $\mu \trianglerighteq \gamma$ if

$$\sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} \gamma_j \text{ for all } i \geq 1.$$

If $\mu \trianglerighteq \gamma$ we say that μ dominates γ. If $\mu \trianglerighteq \gamma$ and $\mu \neq \gamma$ we write $\mu \triangleright \gamma$.

Mathas in [12], gave with a long prove the following two results:

Lemma (2.8): Suppose that $d \in D_\mu \cap D_\gamma^{-1}$, where μ and γ are compositions of r. Then $d^{-1}G_\mu d \cap G_\gamma = G_\tau$ for some τ of r.

Proposition (2.9): Suppose that μ and γ are compositions of r and let

$$D_{\mu\gamma} = \left\{ d \in G_r \left| \begin{array}{l} \gamma(t^\mu d) \text{ is row semi standard and } t^\mu d \trianglerighteq t \\ \text{whenever } t \text{ is a row standard tableau} \\ \text{such that } \gamma(t) = \gamma(t^\mu d) \end{array} \right. \right\}.$$

In fact, $D_{\mu\gamma} = D_\mu \cap D_\gamma^{-1}$

Definition (2.10): On define M^μ to be the right H-module $m_\mu H$, where $m_\mu = \sum_{w \in G_r} T_w$ an element of $H(G_\mu)$.

For the same example after Theorem (2.4), then

$m_\mu = (1+T_1)(1+T_3+T_4+T_3T_4+T_4T_3+T_3T_4T_3)$.

By Lemma (2.8) and Proposition (2.9), we have $\gamma(t^\mu d)$ and $\mu(t^\gamma d^{-1})$ are both row semi-standard and

$$\sum_{\substack{\mu(t^\gamma y)=L \\ y \in D_\gamma}} T_y^* m_\gamma = \sum_{w \in G_\mu d G_\gamma} T_w = \sum_{\substack{\gamma(t^\mu x) \\ x \in D_\mu}} m_\mu T_x. \quad \ldots\ldots\ldots (2.11)$$

Consequently, d determines an H-module homomorphism $\varphi_{\mu\gamma}^d : M^\gamma \to M^\mu$ given by $\varphi_{\mu\gamma}^d(m_\gamma h) = \sum_{w \in G_\mu d G_\gamma} T_w h$ for all $h \in H$. In fact, these elements give a basis of $Hom_H(M^\gamma, M^\mu)$ and this is free as an R-module with basis $\{\varphi_{\mu\gamma}^d \mid d \in D_{\mu\gamma}\}$; see [12, Theorem 4.7].

Dipper and James in [2], define q-Schur algebra by

$$S_q(n,r) = \mathrm{End}_{H_{R,q}}\left(\underset{\mu\in\wedge(n,r)}{\oplus} M^{\mu}\right)$$

where $\wedge(n,r)$ be the set of composition of r with at most n non-zero parts. By definition $S_q(n,r) \cong \underset{\mu,\gamma\in\wedge(n,r)}{\oplus} \mathrm{Hom}_{H_{R,q}}(M^{\gamma},M^{\mu})$, and so q-Schur algebra is free as an R-module with basis $\{\varphi^{d}_{\mu\gamma} \mid d \in D_{\mu\gamma}\}$; see [1 and 2].

3.<u>Solve the problem algebraic and numerical:</u>

From the previous, we see the difficulty of this problem, where the number of probabilities d depends basically on r, so, these probabilities are $d_{r!}$. So, whenever r is large, the computation process will be more complex especially when n is large too.

In this section, we will try to solve this problem algebraic and numerically with a fix n equal 2.

<u>First step</u>

Write the values of γ and μ which are belong to $\wedge(2,r)$ in such table as follow:

		γ				
		(r,o)	(r-1,1)	...	(2,r-2)	(1,r-1)
μ	(r,0)					
	(r-1,1)					
	(2,r-2)					
	(1,r-1)					

...... (3.1)

<u>Second step</u>

As illustrated previously in section (2), the probabilities number d which make $\gamma(t^{\mu}d)$ semi-standard rows when $\gamma=(r,0)$ is $d_{r!}$. Because in any case of μ shown in (3.1), then the numbers that will be appear in the tableaux $\gamma(t^{\mu}d)$ will be holding the digit 1, then they are considered a semi standard rows.

P7 = [1 4 3 2]
P8 = [4 1 3 2]
P9 = [3 1 2 4]
P10 = [3 1 4 2]
P11 = [3 4 1 2]
P12 = [4 3 1 2]
P13 = [2 1 3 4]
P14 = [2 1 4 3]
P15 = [2 4 1 3]
P16 = [4 2 1 3]
P17 = [2 3 1 4]
P18 = [2 3 4 1]
P19 = [2 4 3 1]

P20 = [4 2 3 1]
P21 = [3 2 1 4]
P22 = [3 2 4 1]
P23 = [3 4 2 1]
P24 = [4 3 2 1]

The Group B:
B1 = [4 0]
B2 = [1 3]
B3 = [2 2]
B4 = [3 1]
B5 = [1 1 2]
B6 = [1 2 1]
B7 = [2 1 1]
B8 = [1 1 1 1]

Choose 1 for B1, 2 for B2,,, and so on : 3

A_4PB_3	Semi standards
1 2 3 4	1 1 2 2
1 2 4 3	1 1 2 2
1 4 2 3	Nil

4 1 2 3	Nil
1 3 2 4	Nil
1 3 4 2	1 2 2 1
1 4 3 2	1 2 2 1
4 1 3 2	Nil
3 1 2 4	Nil
3 1 4 2	Nil
3 4 1 2	Nil
4 3 1 2	Nil
2 1 3 4	1 1 2 2
2 1 4 3	1 1 2 2
2 4 1 3	Nil
4 2 1 3	Nil
2 3 1 4	Nil
2 3 4 1	1 2 2 1

2 4 3 1	1 2 2 1
4 2 3 1	Nil
3 2 1 4	Nil
3 2 4 1	Nil
3 4 2 1	Nil
4 3 2 1	Nil

Number of semi standard samples = 8

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