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Modifying Runge – Kutta methods with higher order derivative approximations

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الملخص

نسلط الضوء في هذا البحث إلى بعض طرق رنج – كتا الجديدة والذي يعود الفضل في إيجاده إلى كلا من العلماء [David and Olin] حيث أن ميزة هذه الطرق إنها تتطلب فقط تقييمين للدالة f بالنسبة للطرق ذات الرتبة الثالثة وثلاث تقييمات ل f بالنسبة للطرق ذات الرتبة الرابعة ، إما الطرق ذات الرتبة الخامسة فإنها تتطلب أربع تقييمات ل f ،في حين إن الطرق الاعتيادية تتطلب تقييمات ل f أكثر من ذلك .

وبناءا على الميزة الجديدة لهذه الطرق العددية فقد قمنا بتحسين عمل هذه الطرق من خلال إضافة الحد $\Delta = \gamma y_{n+1} - \gamma y_n \Rightarrow 0$ as $h \to 0$, $n \to \infty$,$0<\gamma<1$ إلى هذه الطرق فكانت النتائج التي حصلنا عليها تشير إلى مدى زيادة دقة عمل هذه الطرق وسرعتها في حل المعادلات.

Abstract

In this paper we modify some sort of Runge-Kutta methods developed by David and Olin which need less function evaluation than ordinary corresponding Runge-Kutta methods. We improve the stability region of these methods by adding a suitable term for the formulas of the methods. We have shown these improvements theoretically and practically.

1-Introduction:-

We introduce the methods developed by David and Olin [5]:

1-1-Third-order method

We consider initial value problems expressed in autonomous from. Starting with the non-autonomous form, we assume that $f(x, y)$ is autonomous function with domain D in R^{n+1} where $x \in R, y \in R^n$, and

$(x,y)\in D$ .we assume that $\| f(x,y_1)-f(x,y_2)\|_2 \le L\|y_1-y_2\|_2$ for all $(x,y_1),(x,y_2)\in D$; thus the problem

$y' = f(x,y)$ (A)

$y(x_0)=y_0$ With $(x_0,y_0)\in D$

Has a unique solution.

In autonomous form, y and f have n+1 components with $y_{n+1}=x$ and $f_{n+1}(y)=1$.

The initial value problem is then written

$y' = f(y)$

$y(x_0)=y_0$ Where $(y_0)_{n+1}=x_0$.

Most efforts to increase the order of the Runge-kutta methods have been accomplished by increasing the number of terms used and thus the number of functional evaluations required [5].

Our method adds higher order derivative terms to the Runge-Kutta K; terms (i>1) to achieve a higher order of accuracy.
For more detail see [2],[3].

For a new third order method, about autonomous systems, David and Olin [5] assume that $y_{n+1}=y_n+b1k1+b2k2$ and $k1=hf(y)$.

They introduce additional terms by assigning

$k_2 = hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1)$, see [1],[5] .

Using Taylor's series expansion techniques, the above is uniquely satisfied to $O(h^3)$ as follows

$k_1 = hf(y_n)$

$k_2 = hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y(y_n)k_1)$

$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{3}{4}k_2$

1-2-Utilizing f_y

The previous section developed a two – stage, third – order method; however, it introduced a term with f_y.

The result is the addition of a higher derivative term to the standard Runge-Kutta method. The following describes three methods to utilize f_y [1].

Method 1: If one knows or can generate f_y. And if the evaluation of f_y, then savings can be realized. For example, with a linear system of equations, $y' = Ay$, is known and constant [1],[5] .

Method 2: since $y'' = f' = f_y f$ for autonomous equations, and since $k_1 = hf$, k_2 can be replaced with

$k_2 = hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y k_1)$

$$= hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y hf)$$

$$= hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f_y f)$$

$$= hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f')$$

Or

$$k_2 = hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 y'')$$

Again, savings can be realized if one can formulate y'' (or f') and if it cheaper to evaluate than f [1],[5] .

Method 3: building onto method2, one approximate y'' (or f') by using the current and previous evaluations of f . For third-order method, this approximation must be of $O(h)$. Since an $O(h)$ approximation of f' is given by $f' = (f_n - f_{n-1})/h$, one can approximate k_2 as follows.

$$k_2 = hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 f')$$

$$= hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h^2 (f_n - f_{n-1})/h)$$

$$= hf(y_n + \frac{2}{3}k_1 + \frac{2}{9}h(f_n - f_{n-1}))$$

Since f_n is calculated in the current step in the evaluation of k_1, one only has to store the previous value f_{n-1}. In effect, the use of previous for the approximation has created a multi step Runge-Kutta method [1],[5] .

Similarly, for fourth - order and fifth - order method for autonomous systems [5].

The new third -, fourth -, and fifth – order numerical integration techniques inspired by the Runge – Kutta method have been presented. The new methods exploit the use of higher order derivatives, specifically f_y . In particular, a technique utilizing an approximation to y'' has been presented resulting in a multistep Runge – Kutta method,

[David and Olin][5] shows the cases where the proposed methods are more efficient that the standard Runge – Kutta methods. Specifically, the proposed methods are more efficient for cases where

. f_y Or y'' is cheaper to evaluate than f .

. The use of historical values of f is cheaper than evaluating f , and

. For the fifth – order case, the number of total functional evaluations can be reduced from 6 to 4 when using an approximation of f' [1].

2- Improving methods

However, the stepsize remains the important key in the performance of these methods, some times using large value for h gives unexpected results.

So that we try to improve the performance of these methods by increasing their stability regions .

Let us consider the general single – step method of order p

$$y_{n+1} - y_n = hQ(x_n + y_n, h) \qquad (1)$$

We always, assume that (1) is consistent and zero stability.

From numerical differentiation, we have

$$y(x_n + h) \approx y(x_n), \text{ as } h \to 0 \qquad (2)$$

Hence, we have $\gamma y(x_n + h) \Rightarrow \gamma y(x_n)$, 0< γ <1, as $h \to 0$.

Therefore (1) may be written as

$$y_{n+1} - y_n + \Delta \approx hQ(x_n + y_n, h) \qquad (3)$$

where $\Delta = \gamma y_{n+1} - \gamma y_n \Rightarrow 0$ as $n \to \infty$, $h \to 0$.

To prove that the improved singlestep methods (3) is convergent, consider the general improved singlestep method (3)

$$(1+\gamma)y_{n+1} - (1+\gamma)y_n \approx hQ(x_n + y_n, h) \qquad (4)$$

Dependence on the definition of consistency (see Lambert [7]) the general improved singlestep methods (3) is said to be consistent with the initial value problems (A), if ([6],[9])

$$Q(x, y, 0) \equiv f(x, y). \qquad (5)$$

If the general improved singlestep method (3) is consistent with IVPs (A), then

$$(1+\gamma)y_{n+1} - (1+\gamma)y_n \approx hQ(x_n + y(x_n), h)$$

$$= hy'(x_n) - hQ(x_n + y(x_n), h) + O(h^2) = O(h^2),$$

where $\gamma hy'(x_n) \Rightarrow 0$ as $h \to 0$, γ is bounded.

For more detail see [4].

So from (5) and dependence on the definition of the order p (see Lambert [7]), the consistent has order at least one.

Therefore by dependence on the definition and theorem of convergence (see [6], [10]), the above methods is convergent if and only if it is consistent .For example ,

Let us consider the improved Runge – Kutta methods

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

Where $k_1 = \dfrac{h}{1+\gamma} f(y_n)$

$$k2 = \frac{h}{1+\gamma} f(y_n + a_{21}k_1 + ha_{22} f_y(y_n)k_1) \qquad (6)$$

For $h \neq 0$ we rewrite the above methods as following

$$\frac{1}{h}(y_{n+1} - y_n) = b_1 k_1 + b_2 k_2$$

Where $k_1 = \frac{1}{1+\gamma} f(y_n)$

$$k2 = \frac{1}{1+\gamma} f(y_n + a_{21}k_1 + ha_{22} f_y(y_n)k_1)$$

Also $\lim_{h\to 0} \frac{(y(x+h) - y(x))}{h} = y' = f$,

We define the truncation error, $T(x,h)$, of (6) to be

$$T(x,h) = \frac{(y(x+h) - y(x))}{h} - b_1 k_1 + b_2 k_2$$

where the $y(x)$ is the solution of (A), and we consider that $T(x,h) \to 0$ as $h \to 0$.

So $\lim_{h\to 0} T(x,h) = \frac{dy}{dx} - Q(x,y,0)$, where $Q(x,y,0) = b_1 k_1 + b_2 k_2$

There for $Q(x,y,0) \equiv f(x,y)$

Thus, from consistency definition we can see that the above method is satisfy the consistency conditions. so that

We get improved Runge – Kutta (IRK3) method which has the following form:

$$y_{n+1} - y_n = \frac{h}{1+\gamma} Q(x_n + y_n, h) \qquad (7)$$

For examples, our new third- fourth- and fifth- order method (IRK) for autonomous system.

2-1 Third-order methods,

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

and $k_1 = \frac{h}{1+\gamma} f(y_n)$,

$$k2 = \frac{h}{1+\gamma} f(y_n + a_{21}k_1 + ha_{22} f_y(y_n)k_1)$$

Using Taylor's series expansion techniques, the above is uniquely satisfied to $O(h^3)$ as follows

$$k_1 = \frac{h}{1+\gamma} f(y_n)$$

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y(y_n)k_1)$$

$$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{2}{9}k_2 .$$

Since $y'' = f' = f_y f$ for autonomous equations, and since $k_1 = \frac{h}{1+\gamma} f(y_n)$, k_2 can be replaced with

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y k_1)$$

$$= \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}hf_y \frac{h}{1+\gamma} f)$$

$$= \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}\frac{h^2}{1+\gamma} f_y f)$$

$$= \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}\frac{h^2}{1+\gamma} f')$$

Building onto methods, since an $O(h)$ approximation of f' is given by $f' = (f_n - f_{n-1})/h$, one can approximate k_2 as follows

$$k_2 = \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}\frac{h^2}{1+\gamma} f')$$

$$= \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

$$= \frac{h}{1+\gamma} f(y_n + \frac{2}{3}k_1 + \frac{2}{9}\frac{h}{1+\gamma}(f_n - f_{n-1}))$$

2-2 Forth-order methods

Now about fourth-order method.

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3$$

where $k_1 = \frac{h}{1+\gamma} f(y_n)$

$$k2 = \frac{h}{1+\gamma} f(y_n + a_{21}k_1 + ha_{22} f_y(y_n)k_1)$$

and

$$k3 = \frac{h}{1+\gamma} f(y_n + a_{31}k_1 + a_{32}k_2 + ha_{33} f_y(y_n)k_1 + ha_{34} f_y(y_n)k_2)$$

However, in order to utilize methods 2 and 3 of Section 1-2, we must restrict the solution with $a_{34} = 0$.

Using Taylor's series expansion techniques, the above is uniquely satisfied to $O(h^3)$ as follows

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{2}k_3$$

$$k_1 = \frac{h}{1+\gamma} f(y_n)$$

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{2}k_1 + \frac{1}{8}hf_y k_1)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - k_1 + 2k_2 - \frac{1}{2} h f_y k_1)$$

Since $y'' = f' = f_y f$ for autonomous equations, and since

$k_1 = \frac{h}{1+\gamma} f(y_n)$, k_2 can be replaced with

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{2}k_1 + \frac{1}{8}\frac{h^2}{1+\gamma} f_y f)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - k_1 + 2k_2 - \frac{1}{2}\frac{h^2}{1+\gamma} f_y f)$$

So, $k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{2}k_1 + \frac{1}{8}\frac{h^2}{1+\gamma} f')$

$$k3 = \frac{h}{1+\gamma} f(y_n - k_1 + 2k_2 - \frac{1}{2}\frac{h^2}{1+\gamma} f')$$

Building onto methods, since an $O(h)$ approximation of f' is given by $f' = (f_n - f_{n-1})/h$, one can approximate k_2 and k_3 as follows

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{2}k_1 + \frac{1}{8}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - k_1 + 2k_2 - \frac{1}{2}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

therefore, $k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{2}k_1 + \frac{1}{8}\frac{h}{1+\gamma}(f_n - f_{n-1}))$

and $k3 = \frac{h}{1+\gamma} f(y_n - k_1 + 2k_2 - \frac{1}{2}\frac{h}{1+\gamma}(f_n - f_{n-1}))$

2-3 Fifth-order methods

For fifth-order method, we have

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4$$

and $k_1 = \frac{h}{1+\gamma} f(y_n)$

$$k2 = \frac{h}{1+\gamma} f(y_n + a_{21}k_1 + h a_{22} f_y(y_n) k_1)$$

$$k3 = \frac{h}{1+\gamma} f(y_n + a_{31}k_1 + a_{32}k_2 + h a_{33} f_y(y_n) k_1)$$

$$k4 = \frac{h}{1+\gamma} f(y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3 + h a_{44} f_y(y_n) k_1)$$

Using Taylor's series expansion techniques, the above is uniquely satisfied to $O(h^3)$ as follows

$$k_1 = \frac{h}{1+\gamma} f(y_n)$$

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{3}k_1 + \frac{1}{18} h f_y k_1)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}hf_y k_1)$$

$$k4 = \frac{h}{1+\gamma} f(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}hf_y k_1)$$

$$y_{n+1} = y_n + \frac{5}{48}k_1 + \frac{27}{56}k_2 + \frac{125}{336}k_3 + \frac{1}{24}k_4$$

Since $y'' = f' = f_y f$ for autonomous equations, and since $k_1 = \frac{h}{1+\gamma} f(y_n)$, k_2 can be replaced with

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{3}k_1 + \frac{1}{18}\frac{h^2}{1+\gamma} f_y f)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}\frac{h^2}{1+\gamma} f_y f)$$

$$k4 = \frac{h}{1+\gamma} f(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}\frac{h^2}{1+\gamma} f_y f)$$

Building onto methods, since an $O(h)$ approximation of f' is given by $f' = (f_n - f_{n-1})/h$, one can approximate k_2, k_3 and k_4 as follows

$$k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{3}k_1 + \frac{1}{18}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

$$k3 = \frac{h}{1+\gamma} f(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

$$k4 = \frac{h}{1+\gamma} f(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}\frac{h^2}{1+\gamma}(f_n - f_{n-1})/h)$$

So, $k2 = \frac{h}{1+\gamma} f(y_n + \frac{1}{3}k_1 + \frac{1}{18}\frac{h}{1+\gamma}(f_n - f_{n-1}))$

$$k3 = \frac{h}{1+\gamma} f(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}\frac{h}{1+\gamma}(f_n - f_{n-1}))$$

$$k4 = \frac{h}{1+\gamma} f(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}\frac{h}{1+\gamma}(f_n - f_{n-1}))$$

3-Numerical results

To demonstrate the new methods are of the order claimed, several equations have been solved using the new third – fourth and fifth – order method.

Here we show by numerical example the performance of the improved methods. From the coming tables we shall show the stability behavior of the normal methods and improved methods by using different values for the stepsize h.

These values, of h showed the stability behavior of the newly developed methods.

We include test results of the following problems for the explicit Runge – Kutta (RK) methods.

Example-1: $y' = -2y + 1$, $y(0) = 1$ and

$$y(x) = \frac{1}{2}\exp(-2x) + \frac{1}{2}$$

Example-2: $y' = -10(y-1)^2$, $y(0) = 2$

$$y(x) = 1 + \frac{1}{1+10x}$$

Example -3: $y' = -0.5y$

$y(0) = 1$, and $y(x) = e^{-0.5x}$

Tables (1) and (2) indicate the effect of step size h which lies outside of the stability region of normal methods and inside for the improved methods. The error columns of Tables (1) and (2) makes it clear that the new methods perform better than the old versions.

Also, from Tables (3), (4)and (5) it can be seen that the improved methods are superior to the normal methods.

Step size = 0.99

X	Theoretical Solution	Numerical Solution of RK Method	Error of RK Method	Numerical Solution of IRK Method	Error of IRK Method
0.99	0.56903	-0.0099	0.57893	-0.0099	0.57893
9.9	0.5	-1.2365	1.7365	0.50003	2.971e-005
19.8	0.5	-0.43413	0.93413	0.5	7.4188e-009
29.7	0.5	0.025937	0.47406	0.5	3.9794e-012
39.6	0.5	0.26017	0.23983	0.5	1.3323e-015
49.5	0.5	0.37869	0.12131	0.5	0
59.4	0.5	0.43864	0.061362	0.5	0
69.3	0.5	0.46896	0.031038	0.5	0
79.2	0.5	0.4843	0.015699	0.5	0
89.1	0.5	0.49206	0.007941	0.5	0
99	0.5	0.49598	0.0040167	0.5	0
108.9	0.5	0.49797	0.0020317	0.5	0
118.8	0.5	0.49897	0.0010277	0.5	0
128.7	0.5	0.49948	0.00051981	0.5	0
138.6	0.5	0.49974	0.00026293	0.5	0

Table (1): Numerical results of EX. –IRK (γ = 0.9) method and RK method .

Step size = 1.5

X	Theoretical Solution	Numerical Solution of RK Method	Error of RK Method	Numerical Solution of IRK Method	Error of IRK Method
1.5	0.52489	1.1875	0.66261	1.1875	0.66261
15	0.5	3272.5	3272	0.50056	0.00055915
30	0.5	4.6984e+007	4.6984e+007	0.5	1.3789e-007
45	0.5	6.7466e+011	6.7466e+011	0.5	1.0627e-011
60	0.5	9.6878e+015	9.6878e+015	0.5	1.7708e-014
75	0.5	1.3911e+020	1.3911e+020	0.5	0
90	0.5	1.9975e+024	1.9975e+024	0.5	0
105	0.5	2.8684e+028	2.8684e+028	0.5	0
120	0.5	4.1188e+032	4.1188e+032	0.5	0
135	0.5	5.9143e+036	5.9143e+036	0.5	0
150	0.5	8.4926e+040	8.4926e+040	0.5	0
165	0.5	1.2195e+045	1.2195e+045	0.5	0
180	0.5	1.7511e+049	1.7511e+049	0.5	0
195	0.5	2.5145e+053	2.5145e+053	0.5	0
210	0.5	3.6107e+057	3.6107e+057	0.5	0

Table (2): Numerical results of EX. –IRK (γ = 0.5) method and RK method .

Step size = 0.12

X	Theoretical Solution	Numerical Solution of RK Method	Error of RK Method	Numerical Solution of IRK Method	Error of IRK Method
0.12	1.4545	1.409	0.045507	1.409	0.045507
1.2	1.0769	-2.0215e+029	2.0215e+029	1.0277	0.049238
2.4	1.04	NaN	NaN	1.0217	0.018318
3.6	1.027	NaN	NaN	1.0178	0.0092083
4.8	1.0204	NaN	NaN	1.0151	0.0052843
6	1.0164	NaN	NaN	1.0131	0.0032564
7.2	1.0137	NaN	NaN	1.0116	0.002087
8.4	1.0118	NaN	NaN	1.0104	0.0013611
9.6	1.0103	NaN	NaN	1.0094	0.00088604
10.8	1.0092	NaN	NaN	1.0086	0.00056258
12	1.0083	NaN	NaN	1.0079	0.00033554
24	1.0041	NaN	NaN	1.0044	0.00027304
36	1.0028	NaN	NaN	1.0031	0.00029627
48	1.0021	NaN	NaN	1.0023	0.00026775
60	1.0017	NaN	NaN	1.0019	0.00023681

Table (3): Numerical results of EX. –IRK ($\gamma = 0.2$) method and RK method .

Step size = 4.5

X	Theoretical Solution	Numerical Solution of RK Method	Error of RK Method	Numerical Solution of IRK Method	Error of IRK Method
4.5	0.1054	0.90137	0.79597	0.90137	0.79597
45	1.6919e-010	0.0024995	0.0024995	1.489e-006	1.4889e-006
90	2.8625e-020	-1.1127e-005	1.1127e-005	1.7456e-013	1.7456e-013
135	4.8431e-030	-3.949e-008	3.949e-008	1.8407e-020	1.8407e-020
180	8.194e-040	1.5984e-011	1.5984e-011	1.9122e-027	1.9122e-027
225	1.3863e-049	3.37e-013	3.37e-013	1.982e-034	1.982e-034
270	2.3456e-059	4.8019e-016	4.8019e-016	2.0537e-041	2.0537e-041
315	3.9684e-069	-1.769e-018	1.769e-018	2.1279e-048	2.1279e-048
360	6.7142e-079	-6.9246e-021	6.9246e-021	2.2047e-055	2.2047e-055
405	1.136e-088	1.3801e-024	1.3801e-024	2.2843e-062	2.2843e-062
450	1.9219e-098	5.6539e-026	5.6539e-026	2.3667e-069	2.3667e-069
900	3.6939e-196	1.5701e-051	1.5701e-051	3.3741e-139	3.3741e-139
1350	7.0995e-294	4.2761e-077	4.2761e-077	4.8103e-209	4.8103e-209
1800	0	1.1391e-102	1.1391e-102	6.8579e-279	6.8579e-279
2250	0	2.9558e-128	2.9558e-128	0	0

Table (4): Numerical results of EX. –IRK (γ = 0.85) method and RK method .

Step size = 2.6

X	Theoretical Solution	Numerical Solution of RK Method	Error of RK Method	Numerical Solution of IRK Method	Error of IRK Method
2.6	0.27253	-0.455	0.72753	-0.455	0.72753
26	2.2603e-006	-0.00064685	0.00064911	-0.00021516	0.00021742
52	5.1091e-012	-5.1591e-006	5.1591e-006	-4.3344e-009	4.3395e-009
78	1.1548e-017	-1.4046e-008	1.4046e-008	-6.9117e-014	6.9129e-014
104	2.6103e-023	-2.756e-011	2.756e-011	-1.0323e-018	1.0323e-018
130	5.9001e-029	-4.1748e-014	4.1748e-014	-1.5079e-023	1.5079e-023
156	1.3336e-034	-4.349e-017	4.349e-017	-2.1851e-028	2.1851e-028
182	3.0144e-040	-4.3251e-021	4.3251e-021	-3.1571e-033	3.1571e-033
208	6.8136e-046	1.2321e-022	1.2321e-022	-4.5566e-038	4.5566e-038
234	1.5401e-051	3.9761e-025	3.9761e-025	-6.5738e-043	6.5738e-043
260	3.4811e-057	8.5192e-028	8.5192e-028	-9.4826e-048	9.4826e-048
520	1.2118e-113	3.9466e-056	3.9466e-056	-3.6954e-096	3.6954e-096
780	4.2184e-170	-7.3971e-083	7.3971e-083	-1.44e-144	1.44e-144
1040	1.4685e-226	-1.531e-110	1.531e-110	-5.6114e-193	5.6114e-193
1300	5.112e-283	4.5598e-138	4.5598e-138	-2.1867e-241	2.1867e-241

Table (5): Numerical results of EX. –IRK (γ = 0.6) method and RK method .

Conclusions

We have improved the method developed by David and Olin by adding the term $\Delta = \gamma y_{n+1} - \gamma y_n \Rightarrow 0$ as $n \to \infty$, $h \to 0$, 0< γ <1 for the method's formulas.

These improvements have been shown theoretically and practically.

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