Existence and Uniqueness Solution of Second Order Nonlinear Integro-Differential Equations with Boundary Conditions

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الملخص

يتضمن البحث دراسة وجود ووحدانية الحل لمعادلات تكاملية–تفاضلية لاخطية من الرتبة الثانية ذات شروط حدودية باستخدام طريقة بيكارد للتقريب المعطاة في المرجع [4] . حيث استطعنا من خلال هذه الدراسة توسيع بعض النتائج المعطاة في المرجع [3] .

ABSTRACT

In this paper we study the existence and uniqueness solution of second order nonlinear integro-differential equations with boundary conditions, by using the Picard approximation method which is given by [4]. Where we extend some results gained by [3].

Introduction

The Picard approximation method has been used to studies in many studies like:

- "Existence and uniqueness solution for certain integro-differential equations", which has the form: [1]

$$\frac{dx}{dt} = f\left(t, x, \int_0^t \phi(s, x(s))ds\right)$$

- "Solutions for the volterra integral equations of the second kind", which has the form: [2]

$$u(x) = f(x) + \int_0^x F(x, y)u(y)dy \quad , \quad (x \in [0, a])$$

$$u(x) = f(x)e^{-\lambda x} + e^{-\lambda x}\int_0^x F(x, y)e^{\lambda y}u(y)dy \quad , \quad (x \in [0, a],\ \lambda > 0)$$

- "picard's successive approximation for non-linear two-point boundary-value problems", which has the form: [3]

$$y'' = f[t, y(t), y'(t)]$$

$$y(a) = A \quad , \quad y(b) = b$$

In this paper we are using the above method for studying the solution of second order nonlinear integro-differential equations with boundary conditions, where we extend some results gained by [3].

Consider the following second order nonlinear integro-differential equations, as the form:

$$\frac{d^2x}{dt^2} = f\left(t, x(t), \dot{x}(t), \int_{-\infty}^{t} w(t,s) g(s, x(s), \dot{x}(s)) ds\right) , \qquad \ldots\ldots (1)$$

with boundary conditions

$$x(a) = A \quad , \quad x(b) = B \qquad \ldots\ldots (2)$$

where the function $f(t, x, \dot{x}, y)$ is a continuous in $t, x, \dot{x}$ and defined on the domain:

$$(t, x, \dot{x}, y) \in [0,T] \times G \times G_1 \times G_2 \qquad \ldots\ldots (3)$$

Where $x \in G \subseteq [0,T]$, $\dot{x} \in G_1 \subseteq [0,T]$ and G_1 , G_2 are closed and bounded domains and A , B are positive constants.

Suppose that the function $f(t, x, \dot{x}, y)$ satisfies the following inequalities:

$$\| f(t, x, \dot{x}, y) \| \le M \quad , \qquad \ldots\ldots (4)$$

$$\| f(t, x_1, \dot{x}_1, y_1) - f(t, x_2, \dot{x}_2, y_2) \| \le K_1 \|x_1 - x_2\| + K_2 \|\dot{x}_1 - \dot{x}_2\| + K_3 \|y_1 - y_2\| , \qquad \ldots\ldots (5)$$

$$\| g(t, x_1, \dot{x}_1) - g(t, x_2, \dot{x}_2) \| \le L_1 \|x_1 - x_2\| + L_2 \|\dot{x}_1 - \dot{x}_2\| \qquad \ldots\ldots (6)$$

for all $t \in [0,T]$ and $x, x_1, x_2 \in G$, $\dot{x}, \dot{x}_1, \dot{x}_2 \in G_1$ and $y, y_1, y_2 \in G_2$, where $M, K_1, K_2, K_3, L_1, L_2,$ are positive constants and $y(t, x, \dot{x}) = \int_{-\infty}^{t} w(t,s) g(s, x(s), \dot{x}(s)) ds$.

The kernel function $w(t,s)$ was defined and continuous in $-\infty < 0 \le a \le \tau \le s \le t \le b \le T < \infty$, $\|w(t,s)\| \le \delta e^{-\gamma(t-s)}$, where γ, δ be a positive constants.

We define the non-empty sets as follows:

$$\left.\begin{aligned} G_f &= G - (T-a)^2 M \\ G_{1f} &= G_1 - (T-a) M \\ G_{2f} &= G_2 - \frac{\delta}{\gamma}\left[L_1 (T-a)^2 - L_2 (T-a)\right] M \end{aligned}\right\} \qquad \ldots\ldots (7)$$

where $\|.\| = \max|.|$.

Furthermore, we suppose that the greatest eigen value of the following matrix:

$$H_0 = \begin{pmatrix} (T-a)^2 q_1 & [1+(T-a)q_2](T-a) \\ (T-a)q_1 & (T-a)q_2 \end{pmatrix}$$

is less than unity. i.e.

$$h_{\max}(H_0) < 1 \qquad \dots \dots (8)$$

Existence Solution

The study of the existence solution of the problem (1), (2) will be introduced by the following:

Theorem 1:

Let the function $f(t,x,\dot{x},y)$ be defined in the domain (3), continuous in $t,x,\dot{x}$ and satisfy the inequalities (4), (5) and (6), then the sequence of functions:

$$x_{m+1}(t,x_0,\dot{x}_0,y_0) = A + (t-a)\dot{x}_m(a) + \int_a^t (t-s)\, f(t, x_m(s), \dot{x}_m(s), y_m(s))ds \qquad \dots \dots (9)$$

with

$$x_0 = A + \frac{(t-a)(B-A)}{(b-a)} \quad , \quad \dot{x}_0(t) = \frac{(B-A)}{(b-a)} \quad , \quad m = 0,1,2,...$$

converges uniformly on the domain:

$$(t,x_0,\dot{x}_0,y_0) \in [0,T] \times G \times G_1 \times G_2 \qquad \dots \dots (10)$$

to the limit function $x_\infty(t,x_0,\dot{x}_0,y_0)$ which is satisfying the integral equation:

$$x(t,x_0,\dot{x}_0,y_0) = A + (t-a)\dot{x}(a) + \int_a^t (t-s)\, f(t, x(s), \dot{x}(s), y(s))ds \qquad \dots \dots (11)$$

provided that:

$$\|x_\infty(t,x_0,\dot{x}_0,y_0) - x_0\| \le (T-a)^2 M \qquad \dots \dots (12)$$

$$\dots \dots (13) \qquad \|\dot{x}_\infty(t,x_0,\dot{x}_0,y_0) - \dot{x}_0\| \le (T-a)M$$

and

$$\begin{pmatrix} \|x_\infty(t,x_0,\dot{x}_0,y_0) - x_m(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}_\infty(t,x_0,\dot{x}_0,y_0) - \dot{x}_m(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix} \le H_0^m (I - H_0)^{-1} \Psi_0 \qquad \dots \dots (14)$$

for $t \in [0,T]$, $x_0 \in G_f$, $\dot{x}_0 \in G_{1f}$, $y_0 \in G_{2f}$ where

$$H_0 = \begin{pmatrix} (T-a)^2 q_1 & [1+(T-a)q_2](T-a) \\ (T-a)q_1 & (T-a)q_2 \end{pmatrix} ,$$

$$\Psi_0 = \begin{pmatrix} (T-a)^2 M \\ (T-a)M \end{pmatrix} \quad , \quad q_1 = K_1 + K_3 \frac{\delta}{\gamma} L_1 \quad , \quad q_2 = K_2 + K_3 \frac{\delta}{\gamma} L_2 .$$

Proof:

Set m=0 and use (9), we get:

$$\|x_1(t, x_0, \dot{x}_0, y_0) - x_0\| = \left\| A + (t-a)\dot{x}_0(a) + \int_a^t (t-s) f(s, x_0, \dot{x}_0, y_0) ds - A - (t-a)\dot{x}_0(a) \right\|$$

$$\leq \int_a^t \|(t-s) f(s, x_0, \dot{x}_0, y_0)\| ds$$

$$= \int_a^t (t-s) \| f(s, x_0, \dot{x}_0, y_0)\| ds \leq (t-a)^2 M$$

$$\|x_1(t, x_0, \dot{x}_0, y_0) - x_0\| \leq (T-a)^2 M \qquad \dots \dots (15)$$

Moreover on differentiating $x_1(t, x_0, \dot{x}_0, y_0)$, we find:

$$\|\dot{x}_1(t, x_0, \dot{x}_0, y_0) - \dot{x}_0\| = \left\| \dot{x}_0 + \int_a^t f(s, x_0, \dot{x}_0, y_0) ds - \dot{x}_0 \right\|$$

$$\leq \int_a^t \| f(s, x_0, \dot{x}_0, y_0)\| ds \leq (t-a)M$$

so that

$$\|\dot{x}_1(t, x_0, \dot{x}_0, y_0) - \dot{x}_0\| \leq (T-a)M \qquad \dots \dots (16)$$

So that from (6), (15) and (16), we find

$$\|y_1(s, x_0, \dot{x}_0) - y_0(s)\| = \left\| \int_{-\infty}^t w(t,s) g(s, x_1(s), \dot{x}_1(s)) ds - \int_{-\infty}^t w(t,s) g(s, x_0, \dot{x}_0) ds \right\|$$

$$\leq \int_{-\infty}^t \|w(t,s)\| \|g(s, x_1(s), \dot{x}_1(s)) - g(s, x_0, \dot{x}_0)\| ds$$

$$\leq \int_{-\infty}^t \delta e^{-\gamma(t-s)} \left[L_1 \|x_1(s) - x_0\| + L_2 \|\dot{x}_1(s) - \dot{x}_0\| \right] ds$$

so

$$\|y_1(s, x_0, \dot{x}_0) - y_0(s)\| \leq \frac{\delta}{\gamma} \left[L_1 (T-a)^2 + L_2 (T-a) \right] M \qquad \dots\dots (17)$$

from (15), (16), (17) and the condition (8), we get $x_1(t,x_0,\dot{x}_0,y_0)\in G$, $\dot{x}_1(t,x_0,\dot{x}_0,y_0)\in G_1$ for all $t\in[0,T]$, $x_0\in G_f$, $\dot{x}_0\in G_{1f}$,

$$y_0(s)=\int_{-\infty}^{t} w(t,s)g(s,x_0,\dot{x}_0)ds\in G_{2f}.$$

Suppose that $x_{m-1}(t,x_0,,\dot{x}_0,y_0)\in G$, $\dot{x}_{m-1}(t,x_0,\dot{x}_0,y_0)\in G_1$, we have

$$\|x_m(t,x_0,\dot{x}_0,y_0)-x_0\|\le (T-a)^2 M$$

$$\|\dot{x}_m(t,x_0,\dot{x}_0,y_0)-\dot{x}_0\|\le (T-a)M$$

$$\|y_m(s,x_0,\dot{x}_0)-y_0(s)\|\le \frac{\delta}{\gamma}\left[L_1(T-a)^2+L_2(T-a)\right]M$$

where $x_1(t,x_0,\dot{x}_0,y_0)\in G$, $\dot{x}_1(t,x_0,\dot{x}_0,y_0)\in G_1$ for all $t\in[0,T]$, $x_0\in G_f$, $\dot{x}_0\in G_{1f}$, $y_0(s)\in G_{2f}$.

We prove now that the sequence (9) is uniformly convergent in (10). From (9), when m=1 we get:

$$\|x_2(t,x_0,\dot{x}_0,y_0)-x_1(t,x_0,\dot{x}_0,y_0)\|=\Big\|A+(t-a)\dot{x}_1(a)+\int_a^t (t-s)f(s,x_1(s),\dot{x}_1(s),y_1(s))ds-$$

$$-A-(t-a)\dot{x}_0(a)-\int_a^t (t-s)f(s,x_0,\dot{x}_0,y_0)ds\Big\|$$

$$\le (t-a)\|\dot{x}_1(a)-\dot{x}_0\|+\int_a^t (t-s)\|f(s,x_1(s),\dot{x}_1(s),y_1(s))-f(s,x_0,\dot{x}_0,y_0)\|ds$$

$$\le (t-a)\|\dot{x}_1(a)-\dot{x}_0\|+\int_a^t (t-s)\left[K_1\|x_1(s)-x_0\|+K_2\|\dot{x}_1(s)-\dot{x}_0\|+K_3\|y_1(s)-y_0\|\right]ds$$

$$\le (t-a)\|\dot{x}_1(a)-\dot{x}_0\|+\int_a^t (t-s)\left[K_1\|x_1(s)-x_0\|+K_2\|\dot{x}_1(s)-\dot{x}_0\|+K_3\frac{\delta}{\gamma}\left(L_1\|x_1(s)-x_0\|+L_2\|\dot{x}_1(s)-\dot{x}_0\|\right)\right]ds$$

$$\le (t-a)\|\dot{x}_1(t)-\dot{x}_0\|+(t-a)^2\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\|+\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_1(t)-\dot{x}_0\|\right]$$

$$=(t-a)^2\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\|+\left[1+(t-a)\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right](t-a)\|\dot{x}_1(t)-\dot{x}_0\|$$

therefore

$$\|x_2(t,x_0,\dot{x}_0,y_0)-x_1(t,x_0,\dot{x}_0,y_0)\|\le (T-a)^2 q_1\|x_1(t)-x_0\|+\left[1+(T-a)q_2\right](T-a)\|\dot{x}_1(t)-\dot{x}_0\|$$

and

$$\left\|\dot{x}_2(t,x_0,\dot{x}_0,y_0)-\dot{x}_1(t,x_0,\dot{x}_0,y_0)\right\|=\left\|\dot{x}_0+\int_a^t f\left(s,x_1(s),\dot{x}_1(s),y_1(s)\right)ds-\dot{x}_0-\int_a^t f\left(s,x_0,\dot{x}_0,y_0\right)ds\right\|$$

$$\leq\int_a^t\left\|f\left(s,x_1(s),\dot{x}_1(s),y_1(s)\right)-f\left(s,x_0,\dot{x}_0,y_0\right)\right\|ds$$

$$\leq\int_a^t\left[K_1\|x_1(s)-x_0\|+K_2\|\dot{x}_1(s)-\dot{x}_0\|+K_3\|y_1(s)-y_0\|\right]ds$$

$$\leq(t-a)\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\|+\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_1(t)-\dot{x}_0\|\right]$$

$$\left\|\dot{x}_2(t,x_0,\dot{x}_0,y_0)-\dot{x}_1(t,x_0,\dot{x}_0,y_0)\right\|\leq(T-a)q_1\|x_1(t)-x_0\|+(T-a)q_2\|\dot{x}_1(t)-\dot{x}_0\|$$

$$\left\|y_2(s,x_0,\dot{x}_0)-y_1(s,x_0,\dot{x}_0)\right\|=\left\|\int_{-\infty}^t w(t,s)g\left(s,x_2(s),\dot{x}_2(s)\right)ds-\int_{-\infty}^t w(t,s)g\left(s,x_1(s),\dot{x}_1(s)\right)ds\right\|$$

$$\leq\int_{-\infty}^t\|w(t,s)\|\left\|g\left(s,x_2(s),\dot{x}_2(s)\right)-g\left(s,x_1(s),\dot{x}_1(s)\right)\right\|ds$$

so

$$\left\|y_2(s,x_0,\dot{x}_0)-y_1(s,x_0,\dot{x}_0)\right\|\leq\frac{\delta}{\gamma}\left[L_1\|x_2(t)-x_1(t)\|+L_2\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

Now when m=2 in (9) we get the following:

$$\left\|x_3(t,x_0,\dot{x}_0,y_0)-x_2(t,x_0,\dot{x}_0,y_0)\right\|=\left\|A+(t-a)\dot{x}_2(a)+\int_a^t(t-s)f\left(s,x_2(s),\dot{x}_2(s),y_2(s)\right)ds-\right.$$

$$\left.-A-(t-a)\dot{x}_1(a)-\int_a^t(t-s)f\left(s,x_1(s),\dot{x}_1(s),y_1(s)\right)ds\right\|$$

$$\leq(t-a)\|\dot{x}_2(a)-\dot{x}_1(a)\|+\int_a^t(t-s)\left\|f\left(s,x_2(s),\dot{x}_2(s),y_2(s)\right)-f\left(s,x_1(s),\dot{x}_1(s),y_1(s)\right)\right\|ds$$

$$\leq(t-a)\|\dot{x}_2(a)-\dot{x}_1(a)\|+\int_a^t(t-s)\left[K_1\|x_2(s)-x_1(s)\|+K_2\|\dot{x}_2(s)-\dot{x}_1(s)\|+K_3\|y_2(s)-y_1(s)\|\right]ds$$

$$\leq(t-a)\|\dot{x}_2(a)-\dot{x}_1(a)\|+\int_a^t(t-s)\left[K_1\|x_2(s)-x_1(s)\|+K_2\|\dot{x}_2(s)-\dot{x}_1(s)\|+K_3\frac{\delta}{\gamma}\left(L_1\|x_2(s)-x_1(s)\|+L_2\|\dot{x}_2(s)-\dot{x}_1(s)\|\right)\right]ds$$

$$\leq(t-a)\|\dot{x}_2(t)-\dot{x}_1(t)\|+(t-a)^2\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\|+\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

$$=(t-a)^2\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\|+\left[1+(t-a)\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right](t-a)\|\dot{x}_2(t)-\dot{x}_1(t)\|$$

$$\left\|x_3(t,x_0,\dot{x}_0,y_0)-x_2(t,x_0,\dot{x}_0,y_0)\right\| \le (T-a)^2 q_1\left\|x_2(t)-x_1(t)\right\|+\left[1+(T-a)q_2\right](T-a)\left\|\dot{x}_2(t)-\dot{x}_1(t)\right\|$$

$$\left\|\dot{x}_3(t,x_0,\dot{x}_0,y_0)-\dot{x}_2(t,x_0,\dot{x}_0,y_0)\right\| = \left\|\dot{x}_0+\int_a^t f(s,x_2(s),\dot{x}_2(s),y_2(s))ds-\dot{x}_0-\int_a^t f(s,x_1(s),\dot{x}_1(s),y_1(s))ds\right\|$$

$$\le \int_a^t \left\|f(s,x_2(s),\dot{x}_2(s),y_2(s))-f(s,x_1(s),\dot{x}_1(s),y_1(s))\right\|ds$$

$$\le \int_a^t \left[K_1\left\|x_2(s)-x_1(s)\right\|+K_2\left\|\dot{x}_2(s)-\dot{x}_1(s)\right\|+K_3\left\|y_2(s)-y_1(s)\right\|\right]ds$$

$$\le (t-a)\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\left\|x_2(t)-x_1(t)\right\|+\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\left\|\dot{x}_2(t)-\dot{x}_1(t)\right\|\right]$$

$$\left\|\dot{x}_3(t,x_0,\dot{x}_0,y_0)-\dot{x}_2(t,x_0,\dot{x}_0,y_0)\right\| \le (T-a)q_1\left\|x_2(t)-x_1(t)\right\|+(T-a)q_2\left\|\dot{x}_2(t)-\dot{x}_1(t)\right\|$$

$$\left\|y_3(s,x_0,\dot{x}_0)-y_2(s,x_0,\dot{x}_0)\right\| = \left\|\int_{-\infty}^t w(t,s)g(s,x_3(s),\dot{x}_3(s))ds-\int_{-\infty}^t w(t,s)g(s,x_2(s),\dot{x}_2(s))ds\right\|$$

$$\le \int_{-\infty}^t \left\|w(t,s)\right\|\left\|g(s,x_3(s),\dot{x}_3(s))-g(s,x_2(s),\dot{x}_2(s))\right\|ds$$

$$\left\|y_3(s,x_0,\dot{x}_0)-y_2(s,x_0,\dot{x}_0)\right\| \le \frac{\delta}{\gamma}\left[L_1\left\|x_3(t)-x_2(t)\right\|+L_2\left\|\dot{x}_3(t)-\dot{x}_2(t)\right\|\right]$$

By induction we have:

$$\left\|x_{m+1}(t,x_0,\dot{x}_0,y_0)-x_m(t,x_0,\dot{x}_0,y_0)\right\| \le (T-a)^2 q_1\left\|x_m(t)-x_{m-1}(t)\right\|+\left[1+(T-a)q_2\right](T-a)\left\|\dot{x}_m(t)-\dot{x}_{m-1}(t)\right\| \quad \dots\dots (18)$$

$$\left\|\dot{x}_{m+1}(t,x_0,\dot{x}_0,y_0)-\dot{x}_m(t,x_0,\dot{x}_0,y_0)\right\| \le (T-a)q_1\left\|x_m(t)-x_{m-1}(t)\right\|+(T-a)q_2\left\|\dot{x}_m(t)-\dot{x}_{m-1}(t)\right\| \quad \dots\dots (19)$$

Rewrite inequalities (18) and (19) in vector from:

$$\Psi_{m+1}(t,x_0,\dot{x}_0,y_0) \le H(t)\ \Psi_m(t,x_0,\dot{x}_0,y_0) \quad \dots\dots (20)$$

$$\Psi_{m+1}(t,x_0,\dot{x}_0,y_0)=\begin{pmatrix}\left\|x_{m+1}(t,x_0,\dot{x}_0,y_0)-x_m(t,x_0,\dot{x}_0,y_0)\right\|\\ \left\|\dot{x}_{m+1}(t,x_0,\dot{x}_0,y_0)-\dot{x}_m(t,x_0,\dot{x}_0,y_0)\right\|\end{pmatrix}$$

$$H(t)=\begin{pmatrix}(t-a)^2 q_1 & \left[1+(t-a)q_2\right](t-a)\\ (t-a)q_1 & (t-a)q_2\end{pmatrix},$$

$$\Psi_m(t,x_0,\dot{x}_0,y_0)=\begin{pmatrix}\left\|x_m(t,x_0,\dot{x}_0,y_0)-x_{m-1}(t,x_0,\dot{x}_0,y_0)\right\|\\ \left\|\dot{x}_m(t,x_0,\dot{x}_0,y_0)-\dot{x}_{m-1}(t,x_0,\dot{x}_0,y_0)\right\|\end{pmatrix}$$

It follows from inequality (20) that:

$$\Psi_{m+1}(t) \le H_0 \ \Psi_m(t) \qquad \text{…… (21)}$$

where

$$H_0 = \max_{t\in[0,T]} H(t)$$

By iterating inequality (21), we have

$$\Psi_{m+1}(t) \le H_0^m \ \Psi_m(t) \qquad \text{…… (22)}$$

where $\Psi_0 = \begin{pmatrix} (T-a)^2 M \\ (T-a)M \end{pmatrix}$

this leads to the estimation:

$$\sum_{i=1}^{m} \Psi_i \le \sum_{i=1}^{m} H_0^{i-1} \ \Psi_0 \qquad \text{…… (23)}$$

since the matrix H_0 has eigenvalues:

$$h_{\max}(H_0) = \frac{1}{2}(T-a)\left([(T-a)q_1 + q_2] + \sqrt{[(T-a)q_1 + q_2]^2 + 4q_1}\right)$$

then the series (9) is uniformly convergent in (10), i.e.

$$\lim_{m\to\infty} \sum_{i=1}^{m} H_0^{i-1} \ \Psi_0 = \sum_{i=1}^{\infty} H_0^{i-1}\Psi_0 = (I - H_0)^{-1} \Psi_0 \qquad \text{…… (24)}$$

where I is identity matrix.

The limiting relation (24) signifies a uniform convergent of the sequence $x_m(t, x_0, \dot{x}_0, y_0)$, $\dot{x}_m(t, x_0, \dot{x}_0, y_0)$

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t, x_0, \dot{x}_0, y_0) &= x_\infty(t, x_0, \dot{x}_0, y_0) \\ \lim_{m\to\infty} \dot{x}_m(t, x_0, \dot{x}_0, y_0) &= \dot{x}_\infty(t, x_0, \dot{x}_0, y_0) \end{aligned}\right\} \qquad \text{……. (25)}$$

By inequality (25), the estimation:

$$\begin{pmatrix} \|x_\infty(t, x_0, \dot{x}_0, y_0) - x_m(t, x_0, \dot{x}_0, y_0)\| \\ \|\dot{x}_\infty(t, x_0, \dot{x}_0, y_0) - \dot{x}_m(t, x_0, \dot{x}_0, y_0)\| \end{pmatrix} \le H_0^m (I - H_0)^{-1} \Psi_0 \qquad \text{…… (26)}$$

is true for m=1,2,3,…

Thus $x_\infty(t, x_0, \dot{x}_0, y_0)$ is a solution of integro-differential equations (1), (2).

Uniqueness solution

The study of the uniqueness solution of the problem (1), (2) will be introduced by the following:

<u>Theorem 2</u>:

Let all assumptions and conditions of theorem 1 be given then the problem (1), (2) has a unique solution $x = x_\infty(t, x_0, \dot{x}_0, y_0)$ on the domain (10).

Proof:

We have to show to that $x(t,x_0,\dot{x}_0,y_0)$ is a unique solution of problem (1), (2). On the contrary, we suppose that there is at least two different solutions $x(t,x_0,\dot{x}_0,y_0)$ and $\hat{x}(t,x_0,\dot{x}_0,y_0)$ of the problem (1), (2).

From (11) the following inequalities are holds:

$$\|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \le (T-a)^2 q_1\|x(t)-\hat{x}(t)\| + [1+(T-a)q_2](T-a)\|\dot{x}(t)-\dot{\hat{x}}(t)\| \quad \dots\dots (27)$$

on differentiating (27) we should also obtain:

$$\|\dot{x}(t,x_0,\dot{x}_0,y_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)\| \le (T-a)q_1\|x(t)-\hat{x}(t)\| + (T-a)q_2\|\dot{x}(t)-\dot{\hat{x}}(t)\| \quad \dots\dots (28)$$

Inequalities (27) and (28) would lead to the estimation:

$$\begin{pmatrix} \|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix} \le H_0 \begin{pmatrix} \|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix}$$

By iterating we should find that:

$$\begin{pmatrix} \|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix} \le H_0^m \begin{pmatrix} \|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix}$$

But $H_0^m \to 0$ as $m \to \infty$, hence proceeding in the last inequality to the limit we should obtain the equalities $x(t,x_0,\dot{x}_0,y_0)=\hat{x}(t,x_0,\dot{x}_0,y_0)$ and $\dot{x}(t,x_0,\dot{x}_0,y_0)=\dot{\hat{x}}(t,x_0,\dot{x}_0,y_0)$ which proves the solution is a unique and this completes the proof of the theorem.

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