Automatic Continuity of Some Types of Double Derivations on Semisimple Banach Algebras

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Abstract
Following Villena in [9] and Mohammed and Ali in [4], we introduce partially defined \((g, h) - c\) - double derivation and generalized \((g, h) - c\) - double derivation on a semisimple complex Banach algebra whose domain is not necessarily closed, essential ideal and we prove that they are closable. In particular, we show that every \((g, h) - c\) -double derivation and generalized \((g, h) - c\) - double derivation defined on any nonzero ideal of a prime \(C^*\) - algebra are continuous.

Keywords: automatic continuity, double derivation, ultraprimness, sliding hump sequence.
0. Introduction

Throughout this paper, \( \mathcal{A} \) is a semisimple Banach algebra over complex field and \( g, h : \mathcal{A} \rightarrow \mathcal{A} \) are linear mappings. If \( g \) and \( h \) are the identity maps and if \( \mathcal{A} \) with or without identity we may conclude that \( g \) and \( h \) are continuous by Johnson and Sinclair in [1]. As a consequence, we can assume that \( g \) and \( h \) are continuous. So, we defined our derivation in this paper as in [5] and [7] as follows: A linear map \( D_1 : \mathcal{A} \rightarrow \mathcal{A} \) is said to be \( (g, h) - c \) - double derivation on \( \mathcal{A} \) if \( D_1(ab) = D_1(a)b + aD_1(b) + g(a)h(b) + h(a)g(b), \forall a, b \in \mathcal{A}. \) Similarly, we defined our derivation in this paper as in [8] as follows: A linear map \( D_2 : \mathcal{A} \rightarrow \mathcal{A} \) is called generalized \( (g, h) - c \) - double derivation on \( \mathcal{A} \) if there exists \( (g, h) - c \) - double derivation \( D_1 : \mathcal{A} \rightarrow \mathcal{A} \) such that \( D_2(ab) = D_2(a)b + aD_1(b) + g(a)h(b) + h(a)g(b), \forall a, b \in \mathcal{A}. \) Recall that, a nonzero ideal \( I \) of \( \mathcal{A} \) is called essential if for any nonzero ideal \( J \) of \( \mathcal{A} \) we have \( I \cap J \neq \{0\}. \) Note that, if \( \mathcal{A} \) is prime then any nonzero ideal of \( \mathcal{A} \) is essential. By essential defined \( (g, h) - c \) - double derivation we mean a linear map \( D_1 : I \rightarrow \mathcal{A} \) such that \( I \) is essential and for all \( a, b \in I, \) \( D_1(ab) = D_1(a)b + aD_1(b) + g(a)h(b) + h(a)g(b). \) Clearly if \( g \) or \( h \) or both are the zero maps then \( D_1 \) is the usual derivation, so \((g,h) - c \) - double derivation is a generalization of derivation. Similarly, by essential defined generalized \((g, h) - c \) - double derivation we mean a linear map \( D_2 : I \rightarrow \mathcal{A} \) such that \( I \) is essential and for all \( a, b \in I, \) \( D_2(ab) = D_2(a)b + aD_1(b) + g(a)h(b) + h(a)g(b). \) Clearly if \( g \) or \( h \) or both are the zero maps and \( D_1 = D_2, \) then \( D_2 \) is the usual derivation, so generalized \((g, h) - c \) - double derivation is a generalization of derivation. Also if \( D_1 = D_2, \) then generalized \((g, h) - c \) - double derivation is \((g, h) - c \) - double derivation.

Automatic continuity of derivations are studied by many researcher, we mention some of them of our present work see [1], [2], [5], [6] and [7].

In this paper, we will follow the same lines of [4] and [9]. We will use \( D = D_1 \) or \( D_2 \) when the results are true for both \( D_1 \) and \( D_2, \) otherwise we will use only \( D_1 \) or \( D_2. \)

Let \( \mathcal{P} \) denote the set of primitive ideals \( P \) of \( \mathcal{A} \) such that \( I \not\in \mathcal{P}. \) The primitive ideal \( P \) can be obtained as the kernel of a continuous irreducible representation of \( \mathcal{A} \) on a complex Banach
space $X_p$, actually the irreducible representation of $\mathcal{A}$ is defined by the following mappings:

$\varphi : \mathcal{A} \rightarrow BL(X_p) \text{ defined by } \varphi(a) = L_a \text{ and } L_a : X_p \rightarrow X_p \text{ defined by } L_a(x) = ax \text{ and the } ker(\varphi) = P \text{ satisfying } \| a \cdot x \| \leq \| a \| \cdot \| x \| \text{ for all } a \in \mathcal{A}, \ x \in X_p.$

Recall that the separating subspace $S(D)$ of $D$ is defined to be the set of those $a$ in $\mathcal{A}$ for which there is a sequence $\{ a_n \}$ in $\mathcal{A}$ with $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} D(a_n) = a$. It is well known that $D$ is closable if and only if $S(D) = 0$, and it is easy to show that $I S(D) + S(D) I \subset S(D)$.

Let $\mathcal{P}_c = \{ P \in \mathcal{P} : S(D) \subset P \}$ and $\mathcal{P}_E = \{ P \in \mathcal{P} : S(D) \not\subset P \}$.

Note that $S(D) \subset \cap_{P \in \mathcal{P}_c} P = P_c$. We will show that $D$ is closed if $P_c = 0$.

1. Main Results

We begin this section by the following fundamental results:

**Proposition 1 :** [9]

Let $P \in \mathcal{P}$ and $J$ any non necessarily closed ideal of $\mathcal{A}$ satisfying $J \not\subset P$. Then one of the following assertions holds:

1) The ideal of those elements $b \in J$ with $\dim bX_p < \infty$ acts irreducibly on $X_p$. Accordingly, given $x, y \in X_p$ with $x \neq 0$ there is $b \in J$ with $\dim bX_p = 1$ and $b \cdot x = y$.

2) There exist sequences $\{ b_n \}$ in $J$ and $\{ x_n \}$ in $X_p$ satisfying $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$ for every $n \in \mathbb{N}$.

Proof : see [9, lemma 1]

Let $\{ P_n \}$ be a sequence in $\mathcal{P}$. A sequence $\{ b_n \}$ in $I$ is said to be a sliding hump sequence for $\{ P_n \}$ if for every $n \in \mathbb{N}$ there exists $x_n \in X_{P_n}$ such that $b_n \ldots b_1 x_n \neq 0$ and $b_{n+1} \ldots b_1 x_n = 0$ (see [9]).

**Proposition 2 :**

If there exists a sliding hump sequence for a sequence $\{ P_n \}$ in $\mathcal{P}$, then there is a natural number $n$ for which

i) $S(D_1) \subset P_n$. In particular, $S(D_1) \subset P$ if $P_n = P$ for every $n \in \mathbb{N}$.

ii) $S(D_2) \subset P_n$. In particular, $S(D_2) \subset P$ if $P_n = P$ for every $n \in \mathbb{N}$.

Proof :
Let \( \{ b_n \} \) be a sliding hump sequence for \( \{ P_n \} \) then for every \( n \in \mathbb{N} \), there exists \( x_n \in X_{P_n} \) such that \( b_n \ldots b_1 x_n \neq 0 \) and \( b_{n+1} \ldots b_1 x_n = 0 \).

We can certainly assume that \( \| b_n \| = \| g \| = \| h \| = \| x_n \| = 1 \) for every \( n \in \mathbb{N} \). We claim that there exist \( n \in \mathbb{N} \) and a nonzero \( x \in X_{P_n} \), such that the map \( a \mapsto D(a)x \) from \( I \) into \( X_{P_n} \) is continuous. If the claim fails, then all the maps \( a \mapsto D(a) b_n \ldots b_1 x_n \) from \( I \) into \( X_{P_n} \) are discontinuous and we can construct inductively a sequence \( \{ a_n \} \) in \( I \) satisfying:

\[
\| D(a_n) b_n \ldots b_1 x_n \| \geq n + \| \sum_{k=1}^{n-1} D(a_k b_k \ldots b_1) x_n \|
\]

\[
+ \| D(c_{n+1}) b_{n+1} \ldots b_1 x_n \| \quad \ldots \quad (1)
\]

and \( \| a_n \| \leq 2^{-n} \min \{ (1 + \| D(b_k b_{k-1} \| )^{-1} : k = 1, \ldots, n \} \).

Now, we consider the element \( c \in \mathcal{A} \) given by \( c = \sum_{n=1}^{\infty} a_n b_n \ldots b_1 \) and for every \( n \in \mathbb{N} \), we write \( c_n = a_n + \sum_{k=n+1}^{\infty} a_k b_k \ldots b_{n+1} \).

Now we will follow the same way of [4] and [9], then we have\( c = \sum_{k=1}^{n-1} a_k b_k \ldots b_1 + a_n b_n \ldots b_1 + c_{n+1} b_{n+1} \ldots b_1 \).

Currently, we will prove the first part of this proposition:

(i) \( D_1(c) = \sum_{k=1}^{n-1} D_1(a_k b_k \ldots b_1) + D_1(a_n) b_n \ldots b_1 + a_n D_1(b_n \ldots b_1) + g(a_n) h(b_n \ldots b_1) + h(a_n) g(b_n \ldots b_1) + D_1(c_{n+1}) b_{n+1} \ldots b_1 + c_{n+1} D_1(b_{n+1} \ldots b_1) + g(c_{n+1}) h(b_{n+1} \ldots b_1) + h(c_{n+1}) g(b_{n+1} \ldots b_1). \)

Now,

\[
\| D_1(c) x_n \| \geq \| D_1(a_n) b_n \ldots b_1 x_n \| - \| \sum_{k=1}^{n-1} D_1(a_k b_k \ldots b_1) x_n \|
\]

\[
- \| a_n D_1(b_n \ldots b_1) x_n \| - \| g(a_n) h(b_n \ldots b_1) x_n \|
\]

\[
- \| h(a_n) g(b_n \ldots b_1) x_n \| - \| D_1(c_{n+1}) b_{n+1} \ldots b_1 x_n \| - \| c_{n+1} D_1(b_{n+1} \ldots b_1) x_n \| - \| g(c_{n+1}) h(b_{n+1} \ldots b_1) x_n \| - \| h(c_{n+1}) g(b_{n+1} \ldots b_1) x_n \| , \quad \text{then by} \ (1) \ \text{we have}
\]

\[
\| D_1(c) x_n \| \geq n - \| a_n D_1(b_n \ldots b_1) x_n \| - \| g(a_n) h(b_n \ldots b_1) x_n \|
\]

\[
- \| h(a_n) g(b_n \ldots b_1) x_n \| - \| c_{n+1} D_1(b_{n+1} \ldots b_1) x_n \| - \| g(c_{n+1}) h(b_{n+1} \ldots b_1) x_n \| \quad \ldots \quad (2)
\]

As a consequence, \( \| a_n D_1(b_n \ldots b_1) x_n \| \leq \| a_n \| \| D_1(b_n \ldots b_1) \| \leq 1 \quad \ldots \quad (3) \)

Also, \( \| g(a_n) h(b_n \ldots b_1) x_n \| \leq \| g \| \| a_n \| \| h \| \| b_n \| \ldots \| b_1 \| \| x_n \| \leq \| a_n \| \leq 1 \quad \ldots \quad (4) \)

Hence, \( \| h(a_n) g(b_n \ldots b_1) x_n \| \leq \| h \| \| a_n \| \| g \| \| b_n \| \ldots \| b_1 \| \| x_n \| \leq \| a_n \| \leq 1 \quad \ldots \quad (5) \)

Now, we will follow the same way of [4] and [9], then we have
Let \( m \in \mathbb{N} \) such that map \( a \mapsto D_1(a)x \) from \( I \) into \( X_{P_m} \) is continuous for some nonzero \( x \in X_{P_m} \) and let \( X \) be the set of all \( x \in X_{P_m} \) satisfying this property, \( X \) is a nonzero \( I \)-submodule of \( X_{P_m} \); therefore, we conclude that \( X = X_{P_m} \). Let \( a \in S(D_1) \) then \( \lim_{n \to \infty} D_1(a_n) = a \) for a suitable sequence \( \{a_n\} \) in \( I \) with \( \lim a_n = 0 \), then \( ax = \lim_{n \to \infty} D_1(a_n)x = 0 \), for every \( x \in X_{P_m} \) and therefore, \( a \in P_m \). That means \( S(D_1) \subset P_m \).

(ii) The proof is similar to the proof of that of first part of this proposition.

**Proposition 3 : [9]**

Let \( P \in \mathcal{P} \) and \( J \) any subspace of \( \mathcal{A} \) satisfying \( IJ +JI \subset J \) and \( J \not\subset P \). Then \( Jx = X_P \) for every nonzero \( x \in X_P \).

**Proof :** see [9, lemma 3]

**Proposition 4 :**

Let \( P \in \mathcal{P} \) and \( J \) any non necessarily closed ideal of \( \mathcal{A} \) contained in \( I \). If there is an element \( b \in J \) such that \( b \not\subset P \), and \( \dim bJb < \infty \).

Then \( S(D_1) \subset P \) and \( S(D_2) \subset P \).
proof:

Note that, since $\dim b/Jb < \infty$ then the map $a \mapsto D(b/Jb)$ is continuous, let $a \in S(D)$, then there exists a sequence $\{a_n\} \subseteq I$ such that $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} D(a_n) = a$. Thus $\lim_{n \to \infty} b a_n b = 0$ and $\lim_{n \to \infty} D(ba_n b) = 0$. Since $g$ and $h$ are continuous linear maps, then $\lim_{n \to \infty} g(a_n) = 0$ and $\lim_{n \to \infty} h(a_n) = 0$, also $\lim_{n \to \infty} b a_n = 0$ thus $\lim_{n \to \infty} g(b a_n) = 0$ and $\lim_{n \to \infty} h(b a_n) = 0$.

Firstly, we will prove $S(D_1) \subseteq P$. Now, for all $b \in I, \{a_n\} \subseteq I$, we have:

$$\lim_{n \to \infty} D_1(b a_n b) = \lim_{n \to \infty} [D_1(ba_n)b + ba_nD_1(b) + g(ba_n)h(b) + h(ba_n)g(b)]$$

$$= \lim_{n \to \infty} [D_1(b) a_n b + b D_1(a_n) b + g(b) h(a_n) b + h(b) g(a_n) b + b a_n D_1(b) + g(b a_n) h(b) + h(b a_n) g(b)]$$

$$= b a b = 0 \ \forall \ a \in S(D_1) \ \text{hence} \ b S(D_1) b = 0$$

Since $b \notin P$ then $b X_P \neq 0$, if we assume that $S(D_1) \notin P$ then by Proposition 3 we have $S(D_1) b X_P = X_P$ thus $b S(D_1) b X_P = b X_P = 0$ Which gives $b \in P$ this is contradiction; therefore, $S(D_1) \subseteq P$.

Secondly, we will prove $S(D_2) \subseteq P$. Since $\lim_{n \to \infty} D_2(a_n) = a$; therefore,

$$\lim_{n \to \infty} b D_2(a_n) = b a \ \text{this implies that} \ \lim_{n \to \infty} D_2(b a_n) = b a,$$

Now, for all $b \in I, \{a_n\} \subseteq I$, we have:

$$\lim_{n \to \infty} D_2(ba_n b) = \lim_{n \to \infty} [D_2(ba_n)b + ba_nD_1(b) + g(ba_n)h(b) + h(ba_n)g(b)]$$

$$= \lim_{n \to \infty} D_2(b a_n) b + \lim_{n \to \infty} b a_n D_1(b) + \lim_{n \to \infty} g(ba_n) h(b) + \lim_{n \to \infty} h(b a_n) g(b)$$

$$= b a b = 0 \ \forall \ a \in S(D_2) \ \text{hence} \ b S(D_2) b = 0$$

Since $b \notin P$ then $b X_P \neq 0$, if we assume that $S(D_2) \notin P$ then by Proposition 3 we have $S(D_2) b X_P = X_P$ then $b S(D_2) b X_P = b X_P = 0$ that means $b \in P$ this is contradiction; therefore, $S(D_2) \subseteq P$.

The proof of the following result may be obtained in the same way as in [9, theorem 5] applying the above propositions 2 and 4.
Proposition 5: \( D_1 \) and \( D_2 \) are closable.

Proof: Obvious.

A Banach algebra \( \mathcal{A} \) is said to be ultraprime if there exists a positive constant \( K \geq 0 \) such that \( K \| a \| \| b \| \leq \| M_{a,b} \| \ \forall a, b \in \mathcal{A} \), where \( M_{a,b} \) is the two-sided multiplication operator on \( \mathcal{A} \) defined by:
\[
M_{a,b}(x) = axb \quad (\text{see} \ [9]).
\]

In [3, proposition 2.3] it was proved that every prime \( C^* \)-algebra is an ultraprime Banach algebra, where \( K = 1 \).

Theorem 6:

Let \( D_1 \) and \( D_2 \) be closable \( (g,h) \)-c- double derivation and generalized \( (g,h) \)-c- double derivation respectively defined on a nonzero ideal 1 of an ultraprime Banach algebra, then \( D_1 \) and \( D_2 \) are continuous.

Proof:

Since \( g \) and \( h \) are continuous; therefore, there are positive constants \( \varepsilon, \delta \geq 0 \) such that \( \| g(y) \| \leq \varepsilon \| y \| \) and \( \| h(z) \| \leq \delta \| z \| \ \forall y, z \in \mathcal{A} \).

Firstly, we will prove \( D_1 \) is continuous. Fix \( a \in I \), with \( \| a \| = 1 \) and consider the following mapping \( f_1: \mathcal{A} \to \mathcal{A} \) define by \( f_1(x) = D_1(xa) \) \( \forall x \in \mathcal{A} \), we will follow the same way of [4] and [9], then we have \( f_1 \) is continuous; therefore, there is a positive constant \( t \geq 0 \) such that \( \| f_1(x) \| \leq t \| x \| \ \forall x \in \mathcal{A} \). Let \( \| x \| = 1 \) we have \( \| f_1(x) \| \leq t \), thus \( \| f_1(x) \| = \| D_1(xa) \| \leq t \). Now, for \( b \in I \), \( x \in \mathcal{A} \) we have:
\[
D_1(bxa) = D_1(b)xa + b D_1(xa) + g(b) h(xa) + h(b) g(xa),
\]
then \( D_1(b)xa = D_1(bxa) - b D_1(xa) - g(b) h(xa) - h(b) g(xa); \) therefore,
\[
M_{D_1(b),a}(x) = D_1(bxa) - b D_1(xa) - g(b) h(xa) - h(b) g(xa),
\]
thus \( \| M_{D_1(b),a}(x) \| \leq \| b \| \| D_1(bxa) \| + \| b \| \| D_1(xa) \| + \| g(b) h(xa) \| + \| h(b) g(xa) \| \leq t \| b \| + \| b \| \| x \| \| a \| \leq 4 \| b \| \| x \| \| a \| \).

By taking supremum for both sides we have \( \| M_{D_1(b),a} \| \leq 4t \| b \| \| a \| \).
Since \( \mathcal{A} \) is ultraprime Banach algebra, then there exists a positive constant
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$K \geq 0$ such that $K \parallel a \parallel \parallel b \parallel \leq \parallel M_{a,b} \parallel$, for all $a, b \in \mathcal{A}$. Then
$K \parallel D_1(b) \parallel \parallel a \parallel \leq \parallel M_{D_1(b),a} \parallel \leq 4t \varepsilon \delta \parallel b \parallel \parallel a \parallel$, hence
$\parallel D_1(b) \parallel \leq \frac{4t \varepsilon \delta}{K} \parallel b \parallel$, $\forall \ b \in I$. This implies that $D_1$ is continuous.

Secondly, we will prove $D_2$ is continuous. Fix $a \in I$, with $\parallel a \parallel = 1$ and consider the following mapping $f_2: \mathcal{A} \to \mathcal{A}$ define by:

$$f_2(x) = D_2(xa) \forall x \in \mathcal{A},$$

we will follow the same way of [4] and [9], then we have $f_2$ is continuous; therefore, there is a positive constant $r \geq 0$, such that

$\parallel f_2(x) \parallel \leq r \parallel x \parallel \forall x \in \mathcal{A}$. Let $\parallel x \parallel = 1$ we have $\parallel f_2(x) \parallel \leq r$, thus

$\parallel f_2(x) \parallel = \parallel D_2(xa) \parallel \leq r$. Now, for $b \in I$, $x \in \mathcal{A}$ we have:

$D_2(bx_a) = D_2(b)x_a + bD_1(xa) + g(b)h(xa) + h(b)g(xa)$, so

$D_2(b)x_a = D_2(bxa) - bD_1(xa) - g(b)h(xa) - h(b)g(xa)$; therefore,

$M_{D_2(b),a}(x) = D_2(bxa) - bD_1(xa) - g(b)h(xa) - h(b)g(xa)$, thus

$\parallel M_{D_2(b),a}(x) \parallel \leq \parallel D_2(bxa) \parallel + \parallel bD_1(xa) \parallel + \parallel g(b)h(xa) \parallel$

$+ \parallel h(b)g(xa) \parallel \leq r + \parallel b \parallel \frac{4t \varepsilon \delta}{K} \parallel xa \parallel + \varepsilon \parallel b \parallel \varepsilon \parallel xa \parallel + \delta \parallel b \parallel \varepsilon \parallel xa \parallel \leq 7rt \varepsilon \delta \parallel b \parallel \parallel a \parallel$.

By taking supremum for both sides we get $\parallel M_{D_2(b),a} \parallel \leq 7rt \varepsilon \delta \parallel b \parallel \parallel a \parallel$. Since $\mathcal{A}$ is ultra prime Banach algebra, then there exists a positive constant $m \geq 0$ such that $m \parallel a \parallel \parallel b \parallel \leq \parallel M_{a,b} \parallel$, for all $a, b \in \mathcal{A}$. Then

$m \parallel D_2(b) \parallel \parallel a \parallel \leq \parallel M_{D_2(b),a} \parallel \leq 7rt \varepsilon \delta \parallel b \parallel \parallel a \parallel$, hence

$\parallel D_2(b) \parallel \leq \frac{7rt \varepsilon \delta}{m} \parallel b \parallel$, $\forall \ b \in I$. This proves that $D_2$ is continuous.

Applying proposition 5 and theorem 6 we can prove the following result:

**Corollary 7:**

Every essentially defined $(g,h)$-c-double derivation and generalized $(g, h)$-c-double derivation on a nonzero ideal of prime $C^*$-algebra is continuous.

**Corollary 8:**

Every essentially defined derivation on a nonzero ideal of prime $C^*$-algebra is continuous.
Proof:
i) By corollary 7, taking \( g \) or \( h \) or both in \( D_1 \) to be the zero maps.
ii) By corollary 7, let \( D_1 = D_2 \) and taking \( g \) or \( h \) or both in \( D_2 \) to be the zero maps.

**Remark 9:**
The above results of this paper are also true for the following derivations:

1) \( D_3 : I \to A \) such that \( D_3(ab) = D_3(a)g(b) + h(a)D_3(b) \), for all \( a, b \in I \).
2) \( D_4 : I \to A \) such that \( D_4(ab) = D_4(a)g(b) + h(a)D_3(b) \), for all \( a, b \in I \).

**References**