حول الدوال المستمرة من النوع-\( i \)

م.م. صبيح وديع اسكندر
قسم الرياضيات/كلية التربية للعلوم الصرفة/جامعة الموصل
الموصل/العراق
sabeehqaqus@yahoo.com

تاريخ الاستلام: 25/02/2018
تاريخ القبول: 07/06/2018

الخلاصة
في هذا البحث نبرهن بأن الدالة \( f : (X, \tau) \rightarrow (Y, \delta) \)
مستمرة من النوع-\( i \) إذا كانت شاملة، متباينة
مستمرة من النوع-\( i \) إلى الفضاء التبولوجي \( (X, \tau) \) من النوع-\( i \)
إضافة إلى ذلك سوف نعرف ونجد العلاقة بين بعض بديهيات الانفصال من النوع-\( i \) مثل \( T_{2i} \) و

الكلمات المفتاحية: الدالة المستمرة من النوع-\( i \)، التراص من النوع-\( i \)، الدالة المفتوحة من النوع-\( i \)
On i-Continuous Functions

Sabih W. Askandar
Department of Mathematics \ College of Education For Pure Science
University of Mosul
Mosul-Iraq
sabeehqaqus@yahoo.com

Received 25/02/2018 Accepted 07/06/2018

ABSTRACT

In this paper we prove that the function \( f : (X, \tau) \rightarrow (Y, \delta) \) is i-open if it is injective, surjective and i-continuous from i-compact topological space \( (X, \tau) \) into \( T_2 \)-space \( (Y, \delta) \). Further, we define and find the relationship among some i-separation axioms such as \( T_{2i}, T_{1i} \) and \( T_{si} \).

Keywords: i-continuous function, i-compactness, i-open function.

INTRODUCTION:

It is well known that a continuous function on topological spaces needs not to be open. In [7] was proved that a continuous function which is injective and surjective from compact topological space \( (X, \tau) \) into \( T_2 \)-space \( (Y, \delta) \). Further, we define and find the relationship among some i-separation axioms such as \( T_{2i}, T_{1i} \) and \( T_{si} \).

Throughout this work \( (\tau, X) \) always are topological spaces (where \( \tau \) is a family of all i-open sets [1] of \( X \)). This work consists of two sections. In the first one we begin by many useful concepts. In the second section we introduce the concept of i-compactness and use it to prove the main result and we have proved some important theorems to discuss the property of i-continuous functions (see Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.9, Theorem 2.10 and Theorem 2.11) and we find the relationship among some i-separation axioms (see Theorem 2.7).
1. Definitions and Examples.

We begin in this section by the following useful concepts.

**Definition 1.1.** A subset $A$ of a topological space $(X, \tau)$ is said to be i-open set[1] if there exists an open set $G \neq \phi$, $X$ such that $A \subseteq Cl(A \cap G).$ The complement of an i-open set is called i-closed set. (Where $Cl(A \cap G)$ denotes the closure of $(A \cap G)$).

**Example 1.2.** Let $X=\{a, b, c, d\}, \tau=\{\phi, \{a\}, \{c, d\}, \{a, c, d\}, X\}$. Open sets are: $\phi, \{a\}, \{c, d\}, \{a, c, d\}, X$ such that if and only if for each $\delta \ni \phi$ containing $\phi$, i.e., $\{a\}, \{c, d\}$. Take $A=\{a, b\}, G=\{a\}, A \cap G=\{a\}, Cl(A \cap G)=\{a, b\}$. So $A \subseteq Cl(A \cap G)$. Then $A$ is i-open set but it is not open. $A^C=\{c, d\}$ is i-closed set.

**Definition 1.3.** Let $(X, \tau')$ be a topological space and let $A$ be a subset of $X$, the intersection of all i-closed sets containing $A$ is called i-Closure of $A[1]$, denoted by $Cl_i(A)$: $Cl_i(A)=\bigcap_{i \ni A} F_i, A \subseteq F_i, \forall i$. Where $F_i$ is i-closed set $\forall i$ in a topological space $(X, \tau_i)$. $Cl_i(A)$ is the smallest i-closed set containing $A$, $A=Cl_i(A)$ if and only if $A$ is i-closed set.

**Definition 1.4.** A function $f: (X, \tau) \rightarrow (Y, \delta)$ is said to be i-continuous[1] at the point $x_0 \in X$ if and only if for each i-open set $I \ni \delta$ containing $f(x_0)$ there exists an i-open set $I$ in $(X, \tau)$ containing $x_0$ such that $f(I) \subseteq I$. $f$ is i-continuous map if it is i-continuous at all points of $X$.

**Lemma 1.5.** [1] Let $f: (X, \tau) \rightarrow (Y, \delta)$ then the following conditions are each equivalent to i-continuity of $f$ on $X$:

i) The inverse of every i-open set in $Y$ is i-open in $X$.

ii) The inverse of every i-closed set in $Y$ is i-closed in $X$.

**Definition 1.6.** A function $f: (X, \tau) \rightarrow (Y, \delta)$ is called i-open [1] if the image $f(O)$ of each i-open set $O$ in $(X, \tau)$ is i-open set in $(Y, \delta)$.

**Definition 1.7.** A topological space $(X, \tau)$ is said to be $T_{ii}$ space, if given any pair of distinct points $x, y$ of $X$, there exists an i-open set $I$, containing one of them but not the other (i-Klomogorov axiom).

**Example 1.8.** Let $X=\{a, b\}, \tau=\{\phi, \{a\}, X\}, \tau'=\{\phi, \{a\}, X\}$ are $(X, \tau')$ and $(X, \tau)$ topological spaces. Therefore: $a \in \{a\}, b \notin \{a\}$ s.t. $a, b \in X (a \neq b) \exists \{a\} \in \tau'$.

**Definition 1.9.** A topological space $(X, \tau)$ is said to be $T_{iii}$ space if for any two distinct points $x, y$ of $X$, there are an i-open set $U$ containing $x$ but not $y$ and i-open set $V$ containing $y$ but not $x$ (i-Frechet axiom).
On i-Continuous Functions

Example 1.10. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$, $\{b, c\}, X \} = \tau'$, $(X, \tau')$ is a topological space.

Let $a, b \in X (a \neq b) \exists\{a\}, \{b\} \in \tau'$

$s.t. \ a \in \{a\}, b \in \{b\}, a \notin \{b\}$

$a, c \in X (a \neq c) \exists\{a\}, \{c\} \in \tau'$

$s.t. \ a \in \{a\}, c \notin \{a\}, c \in \{c\}, a \notin \{c\}$

$b, c \in X (b \neq c) \exists\{b\}, \{c\} \in \tau'$

$s.t. \ b \in \{b\}, c \notin \{b\}, c \in \{c\}, b \notin \{c\}$

Therefore; $(X, \tau')$ is $T_i^{-}$ space.

Definition 1.11. A topological space $(X, \tau)$ is said to be $T_{2i}$ -space if for any two distinct points $x$, $y$ of $X$, there exists two separated i-open sets $I_1$ and $I_2$ such that $I_1$ containing $x$ and $I_2$ containing $y$ (i-Hausdorff axiom).

Example 1.12. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X \}$, $\tau' = \tau$

$(X, \tau')$ and are topological spaces. $(X, \tau')$

$a, b \in X (a \neq b) \exists\{a\}, \{b\} \in \tau'$

$s.t. \ a \in \{a\}, b \in \{b\}, (a \cap \{b\}) = \emptyset$

Therefore; $(X, \tau')$ is $T_{2i}$.

2. i-Compactness and the Main Result.

In this section we introduce the concept of i-compactness and use it to prove the main result also we find the relationship among $T_{2i}$, $T_{i1}$ and $T_{i1}$ -space.

Definition 2.1. A subset $A$ of a topological space $(X, \tau)$ is said to be i-compact if every cover of $A$ by i-open sets has a finite sub cover, we call the topological space $(X, \tau)$ i-compact provided the set $X$ is i-compact.

Theorem 2.2. Every i-closed subset of i-compact space is i-compact.

Proof: Let $A$ be i-closed subset of i-compact space $(X, \tau)$ and let $\{G_k\}$ be a cover of $A$ by i-open sets.

Now $\{G_k \cup A^c\}$ is a cover of $X$ by i-open sets, hence there exists a finite sub cover of $X$. i.e. $X \subseteq \{\cup_{k=1}^{n} G_k \cup A^c\}$ implies to $A \subseteq (\cup_{k=1}^{n} G_k \cup A^c)$. Therefore, $A$ is i-compact.

Theorem 2.3. If $f$ is i-continuous function of $(X, \tau)$ into $(X', \tau')$ then $f$ maps every i-compact subset of $X$ onto an i-compact subset of $X'$.

Proof: Let $A$ be i-compact subset of $X$ and let $\{G_k\}$ be a cover of $f(A)$ by i-open sets.

Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\cup_k G_k^*\right) \subseteq \cup_k f^{-1}(G_k^*)$, the family $f^{-1}(G_k^*)$ is a cover of $A$ by i-open sets by (Theorem 1.5). Since $A$ is i-compact, there must be some finite sub cover of $A$.
say $A \subseteq (\bigcup_{k=1}^{n} f^{-1}(G_k^*) )$. 

Now, $f(A) \subseteq f(\bigcup_{k=1}^{n} f^{-1}(G_k^*)) = \bigcup_{k=1}^{n} f( f^{-1}(G_k^*)) \subseteq \bigcup_{k=1}^{n} G_k^*$. Therefore, $A$ is i-compact.

**Theorem 2.4.** Every i-compact subset $A$ of $T_{2,i}$-space $X$ is i-closed.

**Proof:** Let $x$ be a fixed point in $A^C$. Since the space $X$ is $T_{2,i}$, therefore; for each point $y \in A$, there exist two disjoint i-open sets $G_x$ and $G_y$ such that $x \in G_x$ and $y \in G_y$. The family of sets $\{G_y : y \in A\}$ is a cover of $A$ by i-open sets. Since $A$ is an i-compact, there must be some finite sub covering $\{G_{y_i} : i \in I\}$. Let $\{G_{y_i} : i \in I\}$ be the corresponding i-open sets containing $x$ and let $G = \bigcap_{i=1}^{n} G_y$. Then $G$ is i-open set containing $x$, we see that $G = \bigcap_{i=1}^{n} G_y \subseteq \bigcap_{i=1}^{n} G_y = \bigcap_{i=1}^{n} G_y \subseteq A^C$. Hence each point in $A^C$ is contained in i-open set which is itself contained in $A^C$. Therefore; $A^C$ is i-open set, so $A$ must be i-closed set.

**Theorem 2.5.** Let $f$ be injective, surjective and i-continuous function from i-compact topological space $(X,\tau)$ into $T_{2,i}$-space $(X^*,\tau^*)$ then $f$ is i-open.

**Proof:** Let $A$ be i-open set in $X$, so that $A^C$ is i-closed. By (theorem 2.2), $A^C$ is i-compact. By (theorem 2.3), $f(A^C)$ is i-compact. By (theorem 2.4), $f(A^C)$ is i-closed. Since $f$ is injective and surjective, $f(A^C) = (f(A))^C$. So, as a consequence $f(A)$ is i-open set. Hence $f$ is i-open.

**Example 2.6.** Let $X = \{1,2\}$, $T = \{\emptyset,\{1\},\{2\},X\}$, $Y = \{1,2\}$, $\delta = \{\emptyset,\{1\},\{2\},Y\}$, $f: (X,\tau) \rightarrow (Y,\delta), f(1)=1, f(2)=2, f(3)=3$. It is clear that $(X,\tau)$ is i-compact space because $1,2 \in X$ and $1 \in \{1\}, 2 \in \{2\}$ i.e. $X \subseteq \{1\} \cup \{2\}$, where $\{1\}$ and $\{2\}$ are i-open sets.

So $\{1\} \cup \{2\}$ is a finite i-open cover of $X$.

Also $(Y,\delta)$ is $T_{2,i}$-space because $1,2 \in X$, $1 \neq 2$ there exist two i-open sets $\{1\}$ and $\{2\}$, $\{1\} \cap \{2\} = \emptyset$ such that $1 \in \{1\}, 2 \in \{2\}$.

$f$ is i-continuous function (Theorem 3.3). $f$ is i-open mapping (Definition 3.1).

**Theorem 2.7.** Every $T_{2,i}$-space is $T_{ii}$ and also is $T_{ii}$.

**Proof:** 1. Let $(X,\tau)$ be a $T_{ii}$-space. Let $a$ and $b$ be two distinct points of $(X,\tau)$. Since $(X,\tau)$ is $T_{ii}$-space, there exist two i-open sets $G$ and $H$ such that $a \in G$, $b \notin G$ and $a \notin H, b \in H$. Hence we have $a \in G$, $b \notin G$. Therefore $(X,\tau)$ is $T_{ii}$-space.

2. Let $(X,\tau)$ be a $T_{2,i}$-space. Let $x$ and $y$ be two distinct points in $X$. Since $(X,\tau)$ is $T_{2,i}$-space, there exist two disjoint i-open sets $U$ and $V$ such that $x \in U$, $y \in V$. This implies, $x \in U$, $y \notin U$, $x \notin V$, $y \in V$. Hence $(X,\tau)$ is $T_{ii}$-space.

3. From 1 and 2 we have, every $T_{2,i}$-space is $T_{ii}$. ■
On i-Continuous Functions

The converse of Theorem 2.7 is not necessary to be true. Indeed,

**Example 2.8.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\tau' = \{\emptyset, \{a, b\}, \{a, c\}, X\}$. $(X, \tau')$ is a topological space.

$a, b \in X \ (a \neq b) \ \exists \{a\} \in \tau'$ s.t. $a \in \{a\}, b \notin \{a\}$.

$a, c \in X \ (a \neq c) \ \exists \{a\} \in \tau'$ s.t. $a \in \{a\}, c \notin \{a\}$.

$b, c \in X \ (b \neq c) \ \exists \{a, b\} \in \tau'$ s.t. $b \in \{a, b\}, c \notin \{a, b\}$. Therefore; $(X, \tau)$ is $T_{i1}$. $(X, \tau)$ is not $T_{i2}$-space, because, open $\emptyset$, there is no exist two $i\ a, b \in X \ (a \neq b)$ sets $G$ and $H$ s.t. $a \in G, b \in H$. Also, $(X, \tau)$ is not $T_{i2}$-space (definition 1.11).

**Theorem 2.9.** Let $f : (X, \tau) \to (Y, \delta)$ be bijective i-open and i-continuous map from a $T_{i1}$-space $(X, \tau)$ onto a topological space $(Y, \delta)$. Then $(Y, \delta)$ is $T_{i1}$-space.

**Proof:** Let $a$ and $b$ be two distinct points of $(Y, \delta)$. Since $f$ is bijective, there exist two distinct points $c$ and $d$ of $(X, \tau)$ such that $f(c) = a$ and $f(d) = b$. As $(X, \tau)$ is $T_{i1}$-space, there exists i-open set $G$ such that $c \in G$ and $d \notin G$. Since $f$ is i-open map, then $f(G)$ is i-open in $(Y, \delta)$. As $f$ is i-continuous map we have $a \notin f(G), \ b \notin f(G)$. Hence $(Y, \delta)$ is $T_{i1}$-space.

**Theorem 2.10.** If $f : (X, \tau) \to (Y, \delta)$ is a bijective i-open and i-continuous map from a $T_{i1}$-space $(X, \tau)$ onto a topological space $(Y, \delta)$. Then $(Y, \delta)$ is $T_{i1}$-space.

**Proof:** Let $(X, \tau)$ be a $T_{i1}$-space. Let $a$ and $b$ be two distinct points of $(Y, \delta)$. Since $f$ is bijective, there exist two distinct points $c$ and $d$ of $(X, \tau)$ such that $f(c) = a$ and $f(d) = b$. Since $(X, \tau)$ is $T_{i1}$-space, there exists i-open sets $G$ and $H$ such that $c \in G, d \notin G$ and $c \notin H, d \in H$. Since $f$ is i-open and i-continuous map, then $f(G)$ and $f(H)$ are i-open in $(Y, \delta)$ such that $a = f(c) \in f(G), b = f(d) \notin f(G)$ and $a = f(c) \notin f(H), b = f(d) \in f(H)$. Hence $(Y, \delta)$ is $T_{i1}$-space.

**Theorem 2.11.** Let $f : (X, \tau) \to (Y, \delta)$ be a bijective i-open and i-continuous map. If $(X, \tau)$ is $T_{i2}$-space then $(Y, \delta)$ is also $T_{i2}$-space.

**Proof:** Let $(X, \tau)$ be $T_{i2}$-space. Let $y_1$ and $y_2$ be two distinct points of $Y$. Since $f$ is bijective map, there exist two distinct points $x_1$ and $x_2$ of $X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $(X, \tau)$ is $T_{i2}$-space, there exist i-open sets $G$ and $H$ such that $x_1 \in G, x_2 \in H$ and $G \cap H = \emptyset$. Since $f$ is i-open and i-continuous map, then $f(G)$ and $f(H)$ are i-open in $(Y, \delta)$ such that $y_1 = f(x_1) \in f(G), y_2 = f(x_2) \in f(H)$ and $f(G) \cap f(H) = \emptyset$. Therefore, $f(G) \cap f(H) = f(G \cap H) = \emptyset$. Hence $(Y, \delta)$ is $T_{i2}$-space.

Acknowledgements: Mosul University.
References

(In Arabic)


